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AHP-Based Aczel–Alsina Prioritization under T-Spherical Fuzzy Information for Artificial Intelligence in Smart Systems

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ABSTRACT

This paper presents a novel total prioritized aggregation framework based on the AHP-driven Aczel–Alsina t-norm (AA-TN) and Aczel–Alsina t-conorm (AA-TCN) within the context of T-spherical fuzzy sets (T-SFSs) to address uncertainty in complex decision-making problems. The T-SFS framework provides a flexible and effective environment for representing uncertain information. Furthermore, the AA-TN and AA-TCN offer adaptable operational mechanisms through the parameter ϒ, while prioritized aggregation enables the efficient incorporation of the relative importance of attributes. To integrate these concepts, two novel aggregation models, namely the T-spherical fuzzy Aczel–Alsina averaging (T-SFAAA) and T-spherical fuzzy Aczel–Alsina geometric (T-SFAAG) approaches, are developed. In addition, the analytical hierarchy process (AHP) is employed to determine the priority weights of criteria in multi-attribute decision-making (MADM) problems.

1. Introduction

Given the inherent uncertainty and imprecision associated with decision-analytic problems, Zadeh [1] laid the foundation of fuzzy set (FS) theory. In this framework, a membership grade (MG), restricted to the interval [0,1], is assigned to each element to represent its membership degree to a set. Building upon Zadeh's pioneering work, Atanassov [2] introduced the concept of intuitionistic fuzzy sets (IFSs) by incorporating both membership grade $(b)$and non-membership grade (ϕ), subject to the condition $0 \leq b + \phi \leq 1$, where $b, \phi \in [0,1]$.

To address situations in which $b + \phi > 1$, Yager [3] proposed the concept of Pythagorean fuzzy sets (PyFSs) in 2013, extending the capability of modelling uncertainty through the constraint $0 \leq b^2 + \phi^2 \leq 1$. Subsequently, Yager further generalized this framework by introducing q-rung orthopair fuzzy sets (q-ROFSs), which relax the structural restriction of PyFSs through the condition $0 \leq b^q + \phi^q \leq 1$, where $q \geq 1$. This extension provides a more flexible and comprehensive representation of uncertain information.

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To overcome the limitations of existing models in handling neutrality and abstention, Cuong [4] introduced the concept of picture fuzzy sets (PFSs), an elegant extension of IFSs that incorporates membership grade (b), neutral grade (p), and non-membership grade (ϕ), under the constraint $0 \leq b + p + \phi \leq 1$, where $b, p, \phi \in [0,1]$. By explicitly accounting for indecision and neutrality, PFSs offer a more realistic mathematical framework for complex decision-making problems in which hesitation and non-response play significant roles.

To further enhance the expressive power of fuzzy modelling, Mahmood *et al.*, [5] proposed the notion of spherical fuzzy sets (SFSs), where the admissibility condition is reformulated as $0 \leq b^2 + p^2 + \phi^2 \leq 1$. This spherical constraint enables a broader and more flexible representation of uncertainty, thereby facilitating more nuanced and realistic modelling of practical decision environments. Mahmood *et al.*, [5] subsequently generalized the SFS framework by introducing a tunable power parameter t, leading to the development of T-spherical fuzzy sets (T-SFSs). In this generalized structure, the feasibility condition is expressed as $0 \leq b^t + p^t + \phi^t \leq 1$. By elevating each component to the exponent t, T-SFSs substantially expand the representational capacity of the model, enabling decision-makers to capture varying levels of uncertainty with greater granularity and flexibility. The parameter $t$acts as a control mechanism for adjusting the strictness of the fuzzy sphere, thereby accommodating both conservative and permissive interpretations of expert evaluations.

2. Literature Review

Wei *et al.,* [6] introduced the concept of linguistic aggregation operators (AOs) within the framework of intuitionistic fuzzy sets (IFSs). Nguyen [7] developed novel AOs by integrating estimation techniques and decision-making approaches into the IFS environment. Xu and Liu [8] proposed several geometric aggregation operators based on IFSs. Jabeen *et al.,* [9] extended the notion of Hamacher interactive AOs to the interval-valued Fermatean fuzzy setting. Garg *et al.,* [10] introduced Pythagorean fuzzy aggregation operators based on the Einstein t-norm. Sarkar *et al.,* [11] developed Frank power partitioned Heronian mean AOs for hesitant q-rung orthopair fuzzy sets (q-ROFSs). Mahmood and Ali [12] proposed Hamacher aggregation operators for complex q-ROFSs and demonstrated their applicability in cleaner production assessment for gold mining industries. Senapati *et al.,* [13] investigated Dombi–Archimedean AOs within the q-ROFS framework. Sarfraz [14] applied Maclaurin symmetric mean AOs to spherical fuzzy information using the Schweizer–Sklar t-norm. The authors in [15] conducted a comparative analysis of generalized Einstein aggregation operators in the spherical fuzzy environment. Liu *et al.,* [16] proposed a novel TOPSIS-based decision-making approach utilizing Aczel–Alsina (AA) power Heronian mean operators under the T-spherical fuzzy framework, with applications to pharmaceutical enterprise selection.

Aczel and Alsina [17] pioneered an innovative family of t-norms designed to address uncertainty and incomplete information in fuzzy environments. Senapati *et al.,* [18] introduced AA aggregation tools for IFSs. Asif *et al.,* [19] established AA aggregation operators within the context of complex Pythagorean fuzzy sets and demonstrated their practical applicability. Naeem *et al.,* [20] developed AA geometric aggregation methods for novel picture fuzzy environments. Koam *et al.,* [21] further extended AA aggregation operators to q-ROFSs. Khan *et al.,* [22] proposed a new decision-making framework based on spherical hesitant fuzzy AA geometric operators. Garg *et al.,* [23] developed AA aggregation tools for T-spherical fuzzy information and explored their applications in decision-making problems. Chen *et al.,* [24] investigated machine vision technologies for agricultural applications, while Khatri *et al.,* [25] explored the prioritization of climate-smart agricultural technologies. Winans *et al.,* [26] comparatively analyzed historical and contemporary economic perspectives and proposed new insights into the agricultural industry.

2.1 Research Gaps and Motivations

Although extensive research has been conducted on aggregation operators under various fuzzy environments, including IFSs, PyFSs, q-ROFSs, and SFSs, limited attention has been devoted to the integration of Aczel–Alsina operational laws with prioritized aggregation mechanisms in the T-spherical fuzzy setting. Existing approaches often fail to adequately capture the prioritization relationships among attributes and decision-makers when confronted with highly uncertain and complex information. Moreover, most current MADM frameworks do not incorporate the analytical hierarchy process (AHP) together with AA operational rules under the T-spherical fuzzy environment for intelligent selection and design problems. In particular, there remains a notable research gap in AI-driven smart systems, where uncertainty modelling, ambiguous information processing, and strategic interactions play increasingly significant roles.

2.2 Main Contributions

To address the aforementioned limitations, this study develops a novel AHP-based MADM framework within the T-spherical fuzzy environment. In particular, the Aczel–Alsina t-norm and t-conorm operational laws are integrated with prioritized aggregation mechanisms, resulting in the development of the T-SFAAA and T-SFAAG models. The proposed operators are further examined through rigorous mathematical analysis, where essential properties such as monotonicity, boundedness, and idempotency are formally established. Moreover, an innovative AHP-based MADM methodology is introduced to effectively manage uncertainty and prioritization in complex decision-making problems. The applicability and effectiveness of the proposed framework are ultimately demonstrated through an AI-based smart system case study, thereby confirming its practical relevance and robustness in real-world decision environments.

The remainder of this paper is organized as follows. Section 2 briefly reviews the fundamental concepts related to T-spherical fuzzy sets, AA t-norms, AA t-conorms, and AA operational laws. Section 3 introduces the T-SFAAA and T-SFAAG operators and investigates their essential mathematical properties. In Section 4, the proposed operators are employed to construct a decision-making framework for solving MADM problems under the T-spherical fuzzy environment. Finally, Section 5 concludes the paper and outlines potential directions for future research.

3. Preliminaries

In this section, several fundamental concepts and preliminary notions relevant to the proposed framework are briefly reviewed.

Definition 1: [2] Assuming that F is a universe of discourse. A T-SFS ψ defined in F can be articulated as:

$$\psi = \{(\xi, \mathscr{b}_\psi(\xi), \mathscr{p}_\psi(\xi), \varphi_\psi(\xi) : \xi \in F)\} \quad (1)$$

where $\mathscr{b}_\psi : F \longrightarrow [0,1], \mathscr{p}_\psi : F \longrightarrow [0,1,]$ and $\varphi_\psi : F \longrightarrow [0,1]$ represent MG, NG, and NMG of $\xi \in F$, respectively, bounded by the condition $0 \leq \mathscr{b}_\psi^{\mathfrak{t}}(\xi) + \mathscr{p}_\psi^{\mathfrak{t}}(\xi) + \varphi_\psi^{\mathfrak{t}}(\xi) \leq 1$, for all $\mathfrak{t} \geq 1$. For a T-SFS ψ in F, we represent the hesitancy degree (HD) as:

$$\pi_\psi(\xi) = \sqrt[\mathfrak{t}]{1 - \mathscr{b}_\psi^{\mathfrak{t}}(\xi) - \mathscr{p}_\psi^{\mathfrak{t}}(\xi) - \varphi_\psi^{\mathfrak{t}}(\xi)}, \forall\, \xi \in F \quad (2)$$

For convenience, the triplet $\psi = (\mathscr{b}_\psi, \mathscr{p}_\psi, \varphi_\psi)$ is recognized as a T-Spherical fuzzy value (T-SFV).

Definition 2: [2] Let $\psi_1 = (\mathscr{b}_{\psi_1}, \mathscr{p}_{\psi_1}, \varphi_{\psi_1})$ and $\psi_2 = (\mathscr{b}_{\psi_2}, \mathscr{p}_{\psi_2}, \varphi_{\psi_2})$ be any two T-SFVs in F, then the operational associated to these T-SFVs are postulated follows:

i. $\psi_1 \cup \psi_2 = \left(\max\left(\ell_{\psi_1}, \ell_{\psi_2}\right), \min\left(p_{\psi_1}, p_{\psi_2}\right), \min\left(\varphi_{\psi_1}, \varphi_{\psi_2}\right)\right)$;

ii. $\psi_1 \cap \psi_2 = \left(\min\left(\ell_{\psi_1}, \ell_{\psi_2}\right), \max\left(p_{\psi_1}, p_{\psi_2}\right), \max\left(\varphi_{\psi_1}, \varphi_{\psi_2}\right)\right)$;

iii. $\psi_1 \oplus \psi_2 = \left(\sqrt[t]{\ell^t_{\psi_1} + \ell^t_{\psi_1} - \ell^t_{\psi_1}\ell^t_{\psi_1}}, p_{\psi_1}p_{\psi_2}, \varphi_{\psi_1}\varphi_{\psi_2}\right)$;

iv. $\psi_1 \otimes \psi_2 = \left(\sqrt[t]{\ell^t_{\psi_1}\ell^t_{\psi_2}}, p_{\psi_1} + p_{\psi_2} - p_{\psi_1}p_{\psi_2}, \varphi_{\psi_1} + \varphi_{\psi_2} - \varphi_{\psi_1}\varphi_{\psi_2}\right)$;

v. ${\psi_1}^c = \left(\varphi_{\psi_1}, p_{\psi_1}, \ell_{\psi_1}\right)$;

vi. $\sigma\psi_1 = \left(\sqrt[t]{1 - \left(1 - \ell_{\psi_1}\right)^{\sigma}}, p^{\sigma}_{\psi_1}, \varphi^{\sigma}_{\psi_1}\right)$ for $\sigma \geqslant 0$;

vii. ${\psi_1}^{\sigma} = \left(\sqrt[t]{\ell^{t\sigma}_{\psi_1}}, 1 - \left(1 - p_{\psi_1}\right)^{\sigma}, 1 - \left(1 - \varphi_{\psi_1}\right)^{\sigma}\right)$ for $\sigma \geqslant 0$.

Definition 3: [27] Assume that $\psi_1 = \left(\ell_{\psi_1}, p_{\psi_1}, \varphi_{\psi_1}\right)$ be a T-SFV in F, then score and accuracy functions for ψ_1 are respectively characterized as follows:

$$Sco(\psi_1) = \frac{\left(1 + \ell^{t}_{\psi_1} - p^{t}_{\psi_1} - \varphi^{t}_{\psi_1}\right)}{2} \tag{3}$$

$$Acc\ (\psi_1) = \frac{\left(1 + \ell^{t}_{\psi_1} + p^{t}_{\psi_1} + \varphi^{t}_{\psi_1}\right)}{2} \tag{4}$$

where $Sco(\psi_1) \in [-1, 1]$ and $Sco(\psi_1) \in [0, 1]$.

To determine the order relation of two T-SFVs $\psi_1 = \left(\ell_{\psi_1}, p_{\psi_1}, \varphi_{\psi_1}\right)$ and $\psi_2 = \left(\ell_{\psi_2}, p_{\psi_2}, \varphi_{\psi_2}\right)$, the subsequent results can be utilized:

i. If $Sco(\psi_1) < Sco(\psi_2)$, then $\psi_1 < \psi_2$.
ii. If $Sco(\psi_1) \geqslant Sco(\psi_2)$, then $\psi_1 \geqslant \psi_2$.
iii. If $Sco(\psi_1) = Sco(\psi_2)$, then
   a. If $Acc\ (\psi_1) < Acc\ (\psi_2)$, then $\psi_1 < \psi_2$.
   b. If $Acc\ (\psi_1) \geqslant Acc\ (\psi_2)$, then $\psi_1 \geqslant \psi_2$.
   c. If $Acc\ (\psi_1) = Acc\ (\psi_2)$, then $\psi_1 = \psi_2$.

Definition 4 [27]: A TN is a function $T : [0, 1]^2 \longrightarrow [0, 1]$, such that $\forall\ \ell, v, w, z \in [0, 1]$, the subsequent properties are satisfied:

i. $T(\ell, v) = T(v, \ell)$;
ii. $T(\ell, T(v, w)) = T(T(v, \ell), w)$;
iii. $T(\ell, v) \leq T(w, z)$ if $\ell \leq w$ and $v \leq z$;
iv. $T(1, \ell) = \ell$.

Some examples of TNs are given as:

i. $T_P(\ell, v) = \ell v$ (product TN),
ii. $T_P(\ell, v) = \min\ (\ell, v)$ (minimum TN),
iii. $T_L(\ell, v) = \max\ (\ell + v - 1, 0)$ (Lukasiewicz TN),
iv. $T_D = \begin{cases} \ell, & if\ \ v = 1 \\ v, & if\ \ell = 1 \\ 0, & otherwise \end{cases}$ (drastic TN)

$\forall \ell, v \in [0, 1]$.

Definition 5 [27]: A TCN is a function $S : [0, 1]^2 \longrightarrow [0, 1]$, such that $\forall\ \ell, v, w, z \in [0, 1]$, the following properties are fulfilled:

i. $S(\ell, v) = S(v, \ell)$;
ii. $S(\ell, S(v, w)) = S(S(v, \ell), w)$;
iii. $S(\ell, v) \leq S(w, z)$ if $\ell \leq w$ and $v \leq z$;
iv. $S(0, \ell) = \ell$.

Some illustrations of TNs are given as:

i. $S_P(\ell, v) = \ell + v - \ell v$ (probabilistic sum),
ii. $S_P(\ell, v) = \max(\ell, v)$ (maximum TCN),
iii. $T_L(\ell, v) = \min(\ell + v, 1)$ (Lukasiewicz TCN),
iv. $S_D = \begin{cases} \ell, & if \;\; v = 0 \\ v, & if \;\; \ell = 0 \\ 0, & otherwise \end{cases}$ (drastic TCN)

$\forall \ell, v \in [0, 1]$.

Definition 6 [28]: The AA-TN $(T_A^{\Upsilon})_{\Upsilon \epsilon [0,\infty]}$ is postulated as:

$$(T_A^{\Upsilon})_{(\ell,v)} = \begin{cases} T_D(\ell, v), & if \;\; \Upsilon = 0 \\ min(\ell, v), & if \;\; \Upsilon = \infty \\ e^{-((-ln\ell)^{\Upsilon} + (-lnv)^{\Upsilon})^{\frac{1}{\Upsilon}}}, & \text{otherwise} \end{cases} \tag{5}$$

$\forall \ell, v \in [0, 1]$.

Some particular case: $T_A^{\infty} = min$, $T_A^0 = T_D$, $T_A^1 = T_p$.

Definition 7 [29]: The AA-CTN $(S_A^{\Upsilon})_{\Upsilon \epsilon [0,\infty]}$ is articulated in the following way:

$$(S_A^{\Upsilon})_{(\ell,v)} = \begin{cases} S_D(\ell, v), & if \;\; \Upsilon = 0 \\ max(\ell, v), & if \;\; \Upsilon = \infty \\ 1 - e^{-\left((-ln(1-\ell))^{\Upsilon} + (-ln(1-v))^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, & otherwise \end{cases} \tag{6}$$

Some specific case: $S_A^{\infty} = max$, $S_A^0 = S_D$, $S_A^1 = S_P$.

The TN T_A^{Υ} and TCN S_A^{Υ} are dual to each another for all $\Upsilon \in [0, \infty]$. Besides, the AA-TN is strictly increasing, while the AA-TCN shows a strictly decreasing trend. Moreover, within environment of AA subclass of TNs, they are the only TN family that fully satisfies the equivalence $T(\ell^a, v^a) = T(\ell, v)^a$ for any $a \geqslant 0$ and $\ell, v \in [0,1]$.

Definition 8: Assume that $\psi = (\ell_{\psi}, p_{\psi}, \varphi_{\psi})$, $\psi_1 = (\ell_{\psi_1}, p_{\psi_1}, \varphi_{\psi_1})$ and $\psi_2 = (\ell_{\psi_2}, p_{\psi_2}, \varphi_{\psi_2})$ be any three T-SFVs, $\Upsilon \geq 1$ and $\chi \geqslant 0$, then AA-TN and AA-TCN-based operational rules on them are outlined as:

i. $$\psi_1 \oplus \psi_2 = \begin{pmatrix} \sqrt[\mathfrak{t}]{1 - e^{-\left(\left(-ln\left(1-\ell_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon} + \left(-ln\left(1-\ell_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\left(-ln\left(p_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon} + \left(-ln\left(p_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \\ e^{-\left(\left(-ln\left(\varphi_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon} + \left(-ln\left(\varphi_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}} \end{pmatrix};$$

ii. $$\psi_1 \otimes \psi_2 = \begin{pmatrix} \sqrt[\mathfrak{t}]{e^{-\left(\left(-ln\left(\ell_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon} + \left(-ln\left(\ell_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ 1 - e^{-\left(\left(-ln\left(1-p_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon} + \left(-ln\left(1-p_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \\ 1 - e^{-\left(\left(-ln\left(1-\varphi_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon} + \left(-ln\left(1-\varphi_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}} \end{pmatrix};$$

iii. $$\chi\psi = \left(\begin{array}{c} \sqrt[\mathfrak{t}]{1 - e^{-\left(\chi\left(-ln\left(1-\ell_{\psi}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\chi\left(-ln\left(\wp_{\psi}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\chi\left(-ln\left(\varphi_{\psi}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{array} \right);$$

iv. $$\psi^{\chi} = \left(\begin{array}{c} \sqrt[\mathfrak{t}]{e^{-\left(\chi\left(-ln\left(\ell_{\psi}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ 1 - exp^{-\left(\chi\left(-ln\left(1-\wp_{\psi}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, 1 - exp^{-\left(\chi\left(-ln\left(1-\varphi_{\psi}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{array} \right).$$

Definition 9 [30]: Let $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a gathering of n attributes, and there exists a prioritization between attributes depicted by the linear ordering $\lambda_1 \succ \lambda_2 \succ \cdots \succ \lambda_n$, which indicate attribute λ_i is superior than λ_j when $i < j$. The value $\lambda_i(t)$ specifies the performance of any alternative t regarding the attribute λ_i such that $\lambda_i(t) \in [0,1]$. The formula designated below is said to be the PAO:

$$PAO\left(\lambda_i(t)\right) = \oplus_{i=1}^{n} \mathbb{P}_i \lambda_i(t) \tag{7}$$

where $\mathbb{P}_i = \frac{T_i}{\sum_{i=1}^{\vartheta} T_i}$ with $\mathbb{P}_1 = 1$ and $\mathbb{P}_i = \prod_{\kappa=1}^{i-1} \lambda_\kappa(t)\,; i = 1, 2, \ldots, n$.

4. T-Spherical Fuzzy Aczel-Alsina Aggregation Operators

In this section, we delve into the novel notions concerning T-SFSs to introduce some AA prioritized AOs by adopting the operational laws of AA-TN and AA-TCN, namely T-SFAAPA and T-SFAAPG operators. Also, the validation and vital features and special cases of these newly created operators are explored in detail.

Definition 10: Assume $\psi_\tau = \left(\ell_{\psi_\tau}, \wp_{\psi_\tau}, \varphi_{\psi_\tau}\right)$ $(\tau = 1, 2, \ldots, \vartheta)$ be an array of T-SFVs. Then the T-SFAAPA operator is a function $T - SFAAPA: L^{*\vartheta} \longrightarrow L^{*}$ and is defined as:

$$T - SFAAPA(\psi_1, \psi_2, \ldots \psi_\vartheta) = \oplus_{\tau=1}^{\vartheta} \left(\frac{T_\tau}{\sum_{\tau=1}^{\vartheta} T_\tau}\right) \psi_\tau \tag{8}$$

where $T_\tau = \prod_{\kappa=1}^{\tau-1} Sco(\psi_\kappa)$, $(\tau = 1, 2, \ldots, \vartheta)$, $T_1 = 1$, and $Sco(\psi_\kappa)$ characterizes the score value of kth T-SFV.

Theorem 1: Let $\psi_\tau = \left(\ell_{\psi_\tau}, \wp_{\psi_\tau}, \varphi_{\psi_\tau}\right)$ $(\tau = 1, 2, \ldots, \vartheta)$ be a collection of T-SFVs, then the aggregated value of ψ_τ acquired by using the T-SFAAPA operator is also a T-SFV, and given as:

$$T - SFAAPA(\psi_1, \psi_2, \ldots, \psi_\vartheta) = \left(\begin{array}{c} \sqrt[\mathfrak{t}]{1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau\left(-ln\left(1-\ell_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau\left(-ln\left(\wp_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau\left(-ln\left(\varphi_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}} \end{array} \right) \tag{9}$$

Proof:

We can justify *Theorem 1* using the mathematical induction principle in the following manner:

Step 1: For $\vartheta = 2$, it follows that

$$\omega_1\psi_1 = \begin{pmatrix} \sqrt[\mathfrak{t}]{1 - e^{-\left(\omega_1\left(-ln\left(1-\mathscr{b}^{\mathfrak{t}}_{\psi_1}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\omega_1\left(-ln\left(\wp^{\mathfrak{t}}_{\psi_1}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\omega_1\left(-ln\left(\varphi^{\mathfrak{t}}_{\psi_1}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{pmatrix},$$

$$\omega_2\psi_2 = \begin{pmatrix} \sqrt[\mathfrak{t}]{1 - e^{-\left(\omega_2\left(-ln\left(1-\mathscr{b}^{\mathfrak{t}}_{\psi_2}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\omega_2\left(-ln\left(\wp^{\mathfrak{t}}_{\psi_2}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\omega_2\left(-ln\left(\varphi^{\mathfrak{t}}_{\psi_2}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{pmatrix}.$$

Therefore, according to *Definition 10*, we get

$T - SFAAPA(\psi_1, \psi_2) = \omega_1\psi_1 \oplus \omega_2\psi_2$

$$= \begin{pmatrix} \sqrt[\mathfrak{t}]{1 - e^{-\left(\omega_1\left(-ln\left(1-\mathscr{b}^{\mathfrak{t}}_{\psi_1}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\omega_1\left(-ln\left(\wp^{\mathfrak{t}}_{\psi_1}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\omega_1\left(-ln\left(\varphi^{\mathfrak{t}}_{\psi_1}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{pmatrix} \oplus \begin{pmatrix} \sqrt[\mathfrak{t}]{1 - e^{-\left(\omega_2\left(-ln\left(1-\mathscr{b}^{\mathfrak{t}}_{\psi_2}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\omega_2\left(-ln\left(\wp^{\mathfrak{t}}_{\psi_2}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\omega_2\left(-ln\left(\varphi^{\mathfrak{t}}_{\psi_2}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[\mathfrak{t}]{1 - e^{-\begin{pmatrix} \frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-ln\left(1-\mathscr{b}^{\mathfrak{t}}_{\psi_1}\right)\right)^{\Upsilon} + \\ \frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-ln\left(1-\mathscr{b}^{\mathfrak{t}}_{\psi_2}\right)\right)^{\Upsilon} \end{pmatrix}^{\frac{1}{\Upsilon}}}}, \\ e^{-\begin{pmatrix} \omega_1\left(-ln\left(\wp^{\mathfrak{t}}_{\psi_1}\right)\right)^{\Upsilon} + \\ \omega_2\left(-ln\left(\wp^{\mathfrak{t}}_{\psi_2}\right)\right)^{\Upsilon} \end{pmatrix}^{1/\Upsilon}}, e^{-\begin{pmatrix} \omega_1\left(-ln\left(\varphi^{\mathfrak{t}}_{\psi_1}\right)\right)^{\Upsilon} + \\ \omega_2\left(-ln\left(\varphi^{\mathfrak{t}}_{\psi_2}\right)\right)^{\Upsilon} \end{pmatrix}^{1/\Upsilon}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[\mathfrak{t}]{1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau\left(-ln\left(1-\mathscr{b}^{\mathfrak{t}}_{\psi_\tau}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau\left(-ln\left(\wp^{\mathfrak{t}}_{\psi_\tau}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau\left(-ln\left(\varphi^{\mathfrak{t}}_{\psi_\tau}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{pmatrix}.$$

Hence, Advanced, Eq. (9) is valid for $\vartheta = 2$.

Step 2: Take that Eq. (9) is correct for $\vartheta = k$, then

$$T - SFAAPA(\psi_1, \psi_2, \ldots, \psi_k) = \oplus_{\tau=1}^{k} (\omega_\tau)\psi_\tau$$

$$= \begin{pmatrix} \sqrt[\mathfrak{t}]{1 - e^{-\left(\sum_{\tau=1}^{k} \omega_\tau\left(-ln\left(1-\mathscr{b}^{\mathfrak{t}}_{\psi_\tau}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\sum_{\tau=1}^{k} \omega_\tau\left(-ln\left(\wp^{\mathfrak{t}}_{\psi_\tau}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\sum_{\tau=1}^{k} \omega_\tau\left(-ln\left(\varphi^{\mathfrak{t}}_{\psi_\tau}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{pmatrix}.$$

Step 3: Now, for $\vartheta = k + 1$, we have

$$T - SFAAPA(\psi_1, \psi_2, \ldots, \psi_k, \psi_{k+1}) = \oplus_{\tau=1}^{k} \left((\omega_\tau)\psi_\tau\right) \oplus \left((\omega_{k+1})\psi_{k+1}\right)$$

$$= \left(\left(\begin{array}{c} \sqrt[\mathbb{t}]{1 - e^{-\left(\sum_{\tau=1}^{k} \omega_\tau \left(-ln\left(1-\mathscr{b}_{\psi_\tau}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\sum_{\tau=1}^{k} \omega_\tau \left(-ln\left(\mathscr{p}_{\psi_\tau}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}, e^{-\left(\sum_{\tau=1}^{k} \omega_\tau \left(-ln\left(\varphi_{\psi_\tau}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{array} \right) \oplus \left(\begin{array}{c} \sqrt[\mathbb{t}]{1 - e^{-\left(\omega_{k+1} \left(-ln\left(1-\mathscr{b}_{\psi_{\mathrm{k+1}}}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\omega_{k+1} \left(-ln\left(1-\mathscr{p}_{\psi_{\mathrm{k+1}}}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}, e^{-\left(\omega_{k+1} \left(-ln\left(1-\varphi_{\psi_{\mathrm{k+1}}}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{array} \right) \right)$$

$$= \left(\begin{array}{c} \sqrt[\mathbb{t}]{1 - e^{-\left(\sum_{\tau=1}^{k+1} \omega_{k+1} \left(-ln\left(1-\mathscr{b}_{\psi_\tau}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\sum_{\tau=1}^{k+1} \omega_{k+1} \left(-ln\left(\mathscr{p}_{\psi_\tau}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}, e^{-\left(\sum_{\tau=1}^{k+1} \omega_{k+1} \left(-ln\left(\varphi_{\psi_\tau}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{array} \right)$$

Thus, Eq. (9) is true for $\vartheta = k + 1$, and consequently, in the light of the principle of mathematical induction, the result depicted in Eq. (9) is legitimate for positive integers ϑ. ■

Theorem 1 reveals that the proposed T-SFAAPA operator exhibits the following cardinal characteristics.

Theorem 2 (Idempotency): Presume that $\psi_\tau = \left(\mathscr{b}_{\psi_\tau}, \mathscr{p}_{\psi_\tau}, \varphi_{\psi_\tau}\right)$ $(\tau = 1, 2, \dots, \vartheta)$ be an assortment of T-SFVs, if all $\psi_\tau = \psi = \left(\mathscr{b}_\psi, \mathscr{p}_\psi, \varphi_\psi\right)$ $\forall\, \tau$, then

$$T - SFAAPA(\psi_1, \psi_2, \dots, \psi_\vartheta) = \psi. \tag{10}$$

Proof:

Since $\psi_\tau = \psi$ $\forall\, \tau$, and $\frac{T_\tau}{\sum_{\tau=1}^{\vartheta} T_\tau} = 1$, Therefore, according to *Theorem 2*, it follows that

$$T - SFPAAA(\psi_1, \psi_2, \dots, \psi_\vartheta) = \left(\begin{array}{c} \sqrt[\mathbb{t}]{1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_{k+1} \left(-ln\left(1-\mathscr{b}_{\psi_\tau}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_{k+1} \left(-ln\left(\mathscr{p}_{\psi_\tau}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_{k+1} \left(-ln\left(\varphi_{\psi_\tau}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}} \end{array} \right)$$

$$= \left(\sqrt[\mathbb{t}]{1 - e^{-\left(\left(-ln\left(1-\mathscr{b}_{\psi}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, e^{-\left(\left(-ln\left(\mathscr{p}_{\psi}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\left(-ln\left(\varphi_{\psi}^{\mathbb{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}} \right)$$

$$= \left(\mathscr{b}_\psi, \mathscr{p}_\psi, \varphi_\psi\right)$$

$$= \psi. \qquad ■$$

Theorem 3 (Boundedness): Assume that $\psi_\tau = \left(\mathscr{b}_{\psi_\tau}, \mathscr{p}_{\psi_\tau}, \varphi_{\psi_\tau}\right)(\tau = 1, 2, \dots, \vartheta)$ be a group of T-SFVs, and consider $\psi^- = min(\psi_1, \psi_2, \dots, \psi_\vartheta)$ and $\psi^+ = max(\psi_1, \psi_2, \dots, \psi_\vartheta)$. Then,

$$\psi^- \le T - SFAAPA(\psi_1, \psi_2, \dots, \psi_\vartheta) \le \psi^+. \tag{11}$$

Proof:

Since $\psi^- = min(\psi_1, \psi_2, \dots, \psi_\vartheta) = \left(\mathscr{b}_\psi^-, \mathscr{p}_\psi^-, \varphi_\psi^-\right)$ and $\psi^+ = max(\psi_1, \psi_2, \dots, \psi_\vartheta) = \left(\mathscr{b}_\psi^+, \mathscr{p}_\psi^+, \varphi_\psi^+\right)$, where $\mathscr{b}_\psi^- = \min\{\mathscr{b}_1, \mathscr{b}_2, \dots, \mathscr{b}_\vartheta\}$, $\mathscr{p}_\psi^- = \max\{\mathscr{p}_1, \mathscr{p}_2, \dots, \mathscr{p}_\vartheta\}$, $\varphi_\psi^- =$

$\max\{\varphi_1, \varphi_2, \dots, \varphi_\vartheta\}$, $b_\psi^+ = \max\{b_1, b_2, \dots, b_\vartheta\}$, $p_\psi^+ = \min\{p_1, p_2, \dots, p_\vartheta\}$, and $\varphi_\psi^+ = \min\{\varphi_1, \varphi_2, \dots, \varphi_\vartheta\}$. Thus, the subsequent inequalities exist.

$$\sqrt[\mathfrak{t}]{1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(1-b_\psi^{\mathfrak{t}-}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}} \le \sqrt[\mathfrak{t}]{1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(1-b_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}} \le \sqrt[\mathfrak{t}]{1 - e^{-\left(\sum_{\tau=1} \omega_\tau \left(-ln\left(1-b_\psi^{\mathfrak{t}+}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}. \tag{12}$$

$$e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(p_{\psi_\tau}^{\mathfrak{t}+}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \ge e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(p_\psi^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}} \ge e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(p_{\psi_\tau}^{\mathfrak{t}-}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}. \tag{13}$$

$$e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(\varphi_{\psi_\tau}^{\mathfrak{t}+}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \ge e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(\varphi_\psi^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}} \ge e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(\varphi_{\psi_\tau}^{\mathfrak{t}-}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}. \tag{14}$$

Therefore, by merging inequalities (12) to (14), we obtained $\psi^- \le T - SFAAPA(\psi_1, \psi_2, \dots, \psi_\vartheta) \le \psi^+$. ■

Theorem 4 (Monotonicity): Assume that $\psi_\tau = \left(b_{\psi_\tau}, p_{\psi_\tau}, \varphi_{\psi_\tau}\right)$ and $\psi'_\tau = \left(b'_{\psi_\tau}, p'_{\psi_\tau}, \varphi'_{\psi_\tau}\right)$ be two collections of T-SFVs, where $\tau = 1, 2, \dots, \vartheta$. If $\psi_\tau \le \psi'_\tau \ \forall \ \tau$, then

$$T - SFAAPA(\psi_1, \psi_2, \dots, \psi_\vartheta) \le T - SFAAPA(\psi'_1, \psi'_2, \dots, \psi'_\vartheta). \tag{15}$$

*Proof***:**

As given that $\psi_\tau \le \psi'_\tau \ \forall \ \tau$, therefore $b_{\psi_\tau} \ge b'_{\psi_\tau}$, $p_{\psi_\tau} \le p'_{\psi_\tau}$ and $\varphi_{\psi_\tau} \le \varphi'_{\psi_\tau}$. Based on these facts, we get the following inequalities:

$$\sqrt[\mathfrak{t}]{1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(1-b_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}} \ge \sqrt[\mathfrak{t}]{1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(1-b'^{\mathfrak{t}}_{\psi_\tau}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \tag{16}$$

$$e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(p_\psi^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}} \le e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(p'^{\mathfrak{t}}_{\psi}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \tag{17}$$

$$e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(\varphi_\psi^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}} \le e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(\varphi'^{\mathfrak{t}}_{\psi}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}. \tag{18}$$

Therefore, from inequalities (16) to (18), we have

$$\left(\begin{array}{c} \sqrt[\mathfrak{t}]{1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(1-b_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(p_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(\varphi_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}} \end{array}\right) \le \left(\begin{array}{c} \sqrt[\mathfrak{t}]{1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(1-b'^{\mathfrak{t}}_{\psi_\tau}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_\tau \left(-ln\left(p'^{\mathfrak{t}}_{\psi_\tau}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, e^{-\left(\sum_{\tau=1}^{\vartheta} \frac{T_\tau}{\sum_{\tau=1}^{\vartheta} T_\tau} \left(-ln\left(\varphi'^{\mathfrak{t}}_{\psi_\tau}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}} \end{array}\right).$$

Thus, $T - SFAAPA(\psi_1, \psi_2, \dots, \psi_\vartheta) \le T - SFAAPA(\psi'_1, \psi'_2, \dots, \psi'_\vartheta)$. ■

Theorem 5: Let $\psi_\tau = \left(b_{\psi_\tau}, p_{\psi_\tau}, \varphi_{\psi_\tau}\right)$ $(\tau = 1, 2, \dots, \vartheta)$ be an array of T-SFVs, then the fused outcome determined by employing T-SFAAPG operator is again a T-SFV, and is postulated as:

$$T-SFAAPG(\psi_1,\psi_2,\dots,\psi_\vartheta)=\left(\begin{array}{c} e^{-\left(\sum_{\tau=1}^{\vartheta}\omega_\tau\left(-ln\left(\mathscr{b}_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \\ \sqrt[\mathfrak{t}]{1-e^{-\left(\sum_{\tau=1}^{\vartheta}\omega_\tau\left(-ln\left(1-\mathscr{p}_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ \sqrt[\mathfrak{t}]{1-e^{-\left(\sum_{\tau=1}^{\vartheta}\omega_\tau\left(-ln\left(1-\varphi_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}} \end{array}\right). \quad (19)$$

Proof:
We can prove *Theorem 5* by using the mathematical induction approach on ϑ as follows:
Step 1: When $\vartheta=2$, we have

$$\psi_1{}^{\omega_1}=\left(\begin{array}{c} e^{-\left(\omega_1\left(-ln\left(\mathscr{b}_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_1\left(-ln\left(1-\mathscr{p}_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_1\left(-ln\left(1-\varphi_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}} \end{array}\right)$$

and

$$\psi_2{}^{\omega_2}=\left(\begin{array}{c} e^{-\left(\omega_2\left(-ln\left(\mathscr{b}_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_2\left(-ln\left(1-\mathscr{p}_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_2\left(-ln\left(1-\varphi_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}} \end{array}\right).$$

Therefore,

$$T-SFAAPG(\psi_1,\psi_2)=\psi_1{}^{\omega_1}\otimes\psi_2{}^{\omega_2}$$

$$=\left(\begin{array}{c} e^{-\left(\omega_1\left(-ln\left(\mathscr{b}_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \\ \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_1\left(-ln\left(1-\mathscr{p}_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_1\left(-ln\left(1-\varphi_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}} \end{array}\right)$$

$$\otimes\left(\begin{array}{c} e^{-\left(\omega_2\left(-ln\left(\mathscr{b}_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \\ \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_2\left(-ln\left(1-\mathscr{p}_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_2\left(-ln\left(1-\varphi_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}} \end{array}\right)$$

$$=\left(\begin{array}{c} e^{-\left(\omega_1\left(-ln\left(\mathscr{b}_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon}+\omega_2\left(-ln\left(\mathscr{b}_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \\ \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_1\left(-ln\left(1-\mathscr{p}_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon}+\omega_2\left(-ln\left(1-\mathscr{p}_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_1\left(-ln\left(1-\varphi_{\psi_1}^{\mathfrak{t}}\right)\right)^{\Upsilon}+\omega_2\left(-ln\left(1-\varphi_{\psi_2}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}} \end{array}\right)$$

$$= \left(e^{-\left(\sum_{\tau=1}^{\vartheta}\omega_\tau\left(-ln\left(\mathscr{b}_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\sum_{\tau=1}^{\vartheta}\omega_\tau\left(-ln\left(1-\mathscr{p}_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\sum_{\tau=1}^{\vartheta}\omega_\tau\left(-ln\left(1-\varphi_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}} \right)$$

Therefore, for $\vartheta = 2$, Eq. (19) is true.

Step 2: Presume that Eq. (19) is correct for $\vartheta = k$, then

$$T - SFAAPG(\psi_1, \psi_2, \ldots, \psi_k) = \otimes_{\tau=1}^{k} \psi_\tau^{(\omega_\tau)}$$

$$= \left(e^{-\left(\sum_{\tau=1}^{k}\omega_\tau\left(-l\mathfrak{t}\left(\mathscr{b}_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\sum_{\tau=1}^{k}\omega_\tau\left(-ln\left(1-\mathscr{p}_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\sum_{\tau=1}^{k}\omega_\tau\left(-ln\left(1-\varphi_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}} \right).$$

Step 3: Now, for $\vartheta = k + 1$, we obtain

$$T - SFAAPG(\psi_1, \psi_2, \ldots, \psi_{k+1}) = \otimes_{\tau=1}^{k} \psi_\tau^{(\omega_\tau)} \otimes \psi_{k+1}^{(\omega_{k+1})}$$

$$= \left(e^{-\left(\sum_{\tau=1}^{k}\omega_\tau\left(-ln\left(\mathscr{b}_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\sum_{\tau=1}^{k}\omega_\tau\left(-ln\left(1-\mathscr{p}_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\sum_{\tau=1}^{k}\omega_\tau\left(-ln\left(1-\varphi_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}} \right)$$

$$\otimes \left(e^{-\left(\omega_{k+1}\left(-ln\left(1-\mathscr{b}_{\psi_{k+1}}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_{k+1}\left(-ln\left(1-\mathscr{p}_{\psi_{k+1}}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\omega_{k+1}\left(-ln\left(1-\varphi_{\psi_{k+1}}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}} \right)$$

$$= \left(e^{-\left(\sum_{\tau=1}^{k+1}\omega_\tau\left(-ln\left(\mathscr{b}_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\sum_{\tau=1}^{k+1}\omega_\tau\left(-ln\left(1-\mathscr{p}_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \sqrt[\mathfrak{t}]{1-e^{-\left(\sum_{\tau=1}^{k+1}\omega_\tau\left(-ln\left(1-\varphi_{\psi_\tau}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}}} \right).$$

Thus, Eq. (19) is valid $\vartheta = k + 1$. As a consequence of Steps 1,2 and 3, we conclude that Eq. (19) holds for all positive integers ϑ. ■

The T-SFAAPG operator is likely to satisfy the properties listed below.

Theorem 6 (Idempotency): Assume that $\psi_\tau = \left(\mathscr{b}_{\psi_\tau}, \mathscr{p}_{\psi_\tau}, \varphi_{\psi_\tau}\right)(\tau = 1, 2, \ldots, \vartheta)$ be a collection of T-SFVs. If $\psi_\tau = \psi\ \forall\tau$, then

$$T - SFAAPG(\psi_1, \psi_2, \ldots, \psi_\vartheta) = \psi. \quad (20)$$

Proof:

Analogous to the proof of *Theorem 2*. ■

Theorem 7 (Boundedness): Suppose that $\psi_\tau = \left(\mathscr{b}_{\psi_\tau}, \mathscr{p}_{\psi_\tau}, \varphi_{\psi_\tau}\right)$ $(\tau = 1, 2, \ldots, \vartheta)$ is a family of T-SFVs, and let $\psi^- = min(\psi_1, \psi_2, \ldots, \psi_\vartheta)$ and $\psi^+ = max(\psi_1, \psi_2, \ldots, \psi_\vartheta)$, then

$$\psi^- \leq IFPAAG(\psi_1, \psi_2, \ldots, \psi_\vartheta) \leq \psi^+. \quad (21)$$

Proof:

Similar to the proof of *Theorem 3*. ■

Theorem 8 (Monotonicity): Let $\psi_\tau = \left(\ell_{\psi_\tau}, p_{\psi_\tau}, \varphi_{\psi_\tau}\right)$ and $\psi'_\tau = \left(\ell'_{\psi_\tau}, p'_{\psi_\tau}, \varphi'_{\psi_\tau}\right)$ be two assemblages of T-SFVs, where $\tau = 1, 2, \dots, \vartheta$. If $\psi_\tau \leq \psi'_\tau \ \forall\, \tau$, then

$$T - SFAAPG(\psi_1, \psi_2, \dots, \psi_\vartheta) \leq T - SFAAPG(\psi'_1, \psi'_2, \dots, \psi'_\vartheta). \quad (22)$$

Proof:

Identical the proof of *Theorem 4*. ■

5. MAGDM Method Using T-Spherical Fuzzy Information

The MAGDM process represents an essential decision-support framework for identifying the most suitable alternative from a set of feasible options. The selection procedure is conducted based on the evaluations and judgments provided by a group of experts, who assess the alternatives with respect to multiple relevant attributes. In this section, a novel MAGDM methodology is developed by integrating the proposed aggregation operators with T-spherical fuzzy information in order to demonstrate the reliability, robustness, and effectiveness of the proposed framework.

5.1 Problem Description

Assume that $\mathbb{W} = (\omega_1, \omega_2, \cdots, \omega_n)^T$ denotes the corresponding weight vector of the attributes, satisfying the conditions $0 \leq \omega_\ell \leq 1$ and $\sum_{\ell=1}^{n} \omega_\ell = 1$. To evaluate the alternatives, a group of experts denoted by $\mathbb{D} = \{\mathbb{D}_1, \mathbb{D}_2, \dots, \mathbb{D}_h\}$ is formed, where the experts' weights are represented by $\mathcal{W}_\theta \in [0, 1]$ with $\sum_{\theta=1}^{h} \mathcal{W}_\theta = 1$. These experts assess each alternative $\mathfrak{f}_i (i = 1,2, \dots, m)$ with respect to the corresponding attributes $\mathbb{A}_j (j = 1,2, \dots, n)$. Let $\mathbb{D}_k = (\alpha_{ij}^k)_{n\times m}$ be the decision matrix provided by the k-th expert, where $\alpha_{ij}^k = \left(\ell_{\alpha_{ij}}^k, p_{\alpha_{ij}}^k, \varphi_{\alpha_{ij}}^k\right)$; $(i = 1,2, \dots, m; j = 1,2, \dots, n; k = 1,2, \dots, h)$ denotes the evaluation information expressed in the form of T-spherical fuzzy values (T-SFVs) for alternative $\mathfrak{f}_i$ under the attribute $\mathbb{A}_j$. These values satisfy the condition $0 \leq \left(\ell_{\alpha_{ij}}^k\right)^t + \left(p_{\alpha_{ij}}^k\right)^t + \left(\varphi_{\alpha_{ij}}^k\right)^t \leq 1$ for all $t \geq 1$. Based on the above formulation, the step-by-step procedure of the proposed MADM scheme is systematically presented as follows.

4.2 5.2 Proposed Algorithm

Step 1: Based on the aforementioned analysis, the evaluations of all alternatives with respect to their corresponding attributes are organized in the form of T-spherical fuzzy values (T-SFVs). Subsequently, the decision matrix for each expert is constructed as follows:

$$\mathbb{D}_k = (\alpha_{ij}^k)_{n\times m} = \left(\ell_{\alpha_{ij}}^k, p_{\alpha_{ij}}^k, \varphi_{\alpha_{ij}}^k\right)_{n\times m}. \quad (23)$$

Step 2: For MADM problems, attributes are generally classified into benefit-type and cost-type attributes. When necessary, the T-spherical fuzzy values (T-SFVs) are normalized by transforming all cost-type attributes $(\mathbb{A}_C)$ tinto benefit-type attributes $(\mathbb{A}_B)$. Accordingly, the normalized decision matrices $\aleph_k = (\widetilde{\alpha_{ij}^k})_{n\times m}$ are obtained using the following transformation

$$\widetilde{\alpha_{ij}^k} = \begin{cases} \alpha_{ij}^k;\ j \in \mathbb{A}_B \\ (\widetilde{\alpha_{ij}^k})^c;\ j \in \mathbb{A}_C, \end{cases} \quad (24)$$

where $(\widetilde{\alpha_{ij}^k})^c = \left(\varphi_{\alpha_{ij}}^k,\ p_{\alpha_{ij}}^k,\ \ell_{\alpha_{ij}}^k\right)$ denotes the complement of α_{ij}^k.

Step 3: Assume that $K^{\mathfrak{t}} = (K_{\iota\tau}^{\mathfrak{t}})_{m\times n}$ is a T-spherical fuzzy decision matrix, where $K_{\iota\tau}^{\mathfrak{t}} = (\ell_{\iota\tau}, p_{\iota\tau}, \varphi_{\iota\tau})$ denotes the attribute value provided by the decision-maker $exp_{\mathfrak{t}}$. This evaluation is expressed in the form of a T-spherical fuzzy value (T-SFV).

$$r_{\iota\tau}^{\mathfrak{t}} = \begin{cases} k_{\iota\tau}^{\mathfrak{t}}, for \text{ benefit attributes } c_\tau \\ k_{\iota\tau}^{\mathfrak{t}}, for \text{ cost attributes } c_\tau \end{cases}$$

Next, the T-SFAAA operator is employed to develop an MADM approach within the T-spherical fuzzy environment. The main steps of the proposed procedure are presented as follows:

Step 1: Using the following equations as a basis, the values of $T_{\iota\tau}^{\mathfrak{t}}$, where $\mathfrak{t} = 1,2,\ldots,p$, are determined.

$$T_{\tau=}\prod_{k=1}^{\tau-1} S\left(r_{\iota\tau}^{\mathfrak{t}}\right)(\mathfrak{t}=2,\ldots p), T_{\iota\tau}^{1}=1$$

Step 2: Apply the T-SFAAA operator, defined as follows:

$$r_{\iota\tau} = T - SFAAA\left(\psi_{\iota\tau}{}^{1}, \psi_{\iota\tau}{}^{2}, \ldots, \psi_{\iota\tau}{}^{p}{}_{\vartheta}\right) = \begin{pmatrix} \sqrt[\mathfrak{t}]{1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_{\tau}{}^{(p)}\left(-ln\left(1-\mathscr{b}_{\psi_{\tau}}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_{\tau}{}^{(p)}\left(-ln\left(\mathscr{p}_{\psi}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \\ e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_{\tau}{}^{(p)}\left(-ln\left(\varphi_{\psi}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{pmatrix}$$

or the T-SFAAG operator, defined as follows:

$$r_{\iota\tau} = T - SFAAG\left(\psi_{\iota\tau}{}^{1}, \psi_{\iota\tau}{}^{2}, \ldots \psi_{\iota\tau}{}^{p}{}_{\vartheta}\right) = \begin{pmatrix} \sqrt[\mathfrak{t}]{e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_{\tau}{}^{(p)}\left(-ln\left(\mathscr{b}_{\psi_{\tau}}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}}, \\ 1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_{\tau}{}^{(p)}\left(-ln\left(1-\mathscr{p}_{\psi}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{\frac{1}{\Upsilon}}}, \\ 1 - e^{-\left(\sum_{\tau=1}^{\vartheta} \omega_{\tau}{}^{(p)}\left(-ln\left(1-\varphi_{\psi}^{\mathfrak{t}}\right)\right)^{\Upsilon}\right)^{1/\Upsilon}} \end{pmatrix}$$

To construct the collective T-spherical fuzzy decision matrix $R^{\mathfrak{t}} = \left(r_{\iota\tau}^{\mathfrak{t}}\right)_{m\times n}(\mathfrak{t} = 2,\ldots p)$ all individual T-spherical fuzzy decision matrices $R = (r_{\iota\tau})_{m\times n}$ are aggregated.

Step 3: Determine $T_{\iota\tau}$, $(\iota = 1,2,\ldots m, \tau = 1,2,\ldots \mathfrak{t})$, by solving the following equations:

$$T_{\tau=}\prod_{k=1}^{\tau-1} S\left(r_{\iota\tau}^{\mathfrak{t}}\right)(\mathfrak{t}=2,\ldots p), T_{\iota\tau}^{1}=1, T_{\iota\tau}^{1}=1.$$

Step 4: Using the score function defined in Section 2, rank the alternatives as follows:

$$Sco(\psi_{\iota}) = (1 + \mathscr{b}_{\psi_{\tau}}^{\mathfrak{t}} - \mathscr{p}_{\psi}^{\mathfrak{t}} - \varphi_{\psi}^{\mathfrak{t}})/2 \;\; \iota = 1,2,\ldots m$$

Accordingly, for each alternative k_{ι}, $(\iota = 1,2,\ldots,m)$, with the corresponding overall T-spherical fuzzy value r_{ι}, a higher value of $S(r_{\iota})$indicates a more preferable alternative.

5.3 Analytic Hierarchy Process (AHP) Method

The AHP methodology is conducted with the help of Saaty, used to solve issues of MADM, where multiple options are evaluated under multiple criteria to determine major options in an established way that is widely applied in practice in areas such as banking, supplier identification, threat assessment, and financial capability.

Step 1. Define the problem and establish the hierarchy. The decision problem is structured into three hierarchical levels: the overall goal (G), the evaluation criteria $(C_1, C_2, \ldots, C_n)$, and the set of alternatives.

Step 2. Construct the pairwise comparison matrix. The decision-makers compare the criteria using Saaty's 1–9 scale. The pairwise comparison matrix is expressed as

$$A = (a_{ij})_{n\times n},$$

where a_{ij}represents the relative importance of criterion $C_i$over criterion C_j. The matrix satisfies

$$a_{ij} > 0, a_{ii} = 1, a_{ji} = \frac{1}{a_{ij}}.$$

Step 3. Normalize the pairwise comparison matrix. Each element of the matrix is normalized as follows:

$$\bar{a}_{ij} = \frac{a_{ij}}{\sum_{i=1}^{n} a_{ij}}.$$

Step 4. Calculate the priority weights. The priority weight of each criterion is obtained by averaging the elements in the corresponding row of the normalized matrix:

$$w_i = \frac{1}{n}\sum_{j=1}^{n} \bar{a}_{ij}, i = 1,2, \dots, n.$$

Step 5. Conduct the consistency test. The consistency index is calculated as

$$CI = \frac{\lambda_{\max} - n}{n - 1},$$

where $\lambda_{\max}$denotes the maximum eigenvalue of the pairwise comparison matrix.

Step 6. Calculate the consistency ratio. The consistency ratio is obtained as

$$CR = \frac{CI}{RI}.$$

If $CR < 0.10$, the pairwise comparisons are considered acceptable; otherwise, the judgments should be revised.

6. Practical Example

Artificial intelligence (AI) has become one of the most influential technologies in modern society, transforming various sectors, including healthcare, education, transportation, finance, and cybersecurity. AI enhances organizational performance, automates complex tasks, and supports intelligent decision-making. For instance, AI-powered healthcare systems can assist in disease detection through medical imaging, while intelligent transportation systems can optimize traffic flow and reduce fuel consumption. Similarly, AI-based educational platforms provide personalized learning experiences, and AI-driven business analytics improve operational efficiency.

Let $\Phi_1, \Phi_2, \dots, \Phi_5$denote the AI-based alternatives considered in this study:

Φ_1 - Smart healthcare system: An AI-based healthcare platform used for disease prediction, patient monitoring, and medical decision support.

Φ_2 - Smart transportation system: An AI-powered transportation management system designed for traffic control, route optimization, and accident reduction.

Φ_3 - AI-based learning platform: An intelligent learning system that provides personalized educational content and automated assessment.

Φ_4 - Financial fraud detection system: AI software used by banks and financial institutions to identify suspicious transactions and reduce fraudulent activities.

Φ_5 - Smart security and monitoring system: An AI-powered surveillance system capable of facial recognition, motion detection, and risk assessment.

These alternatives represent AI-based software systems that improve efficiency, automate processes, and enable intelligent decision-making in real-world environments. The alternatives are evaluated with respect to the following criteria:

$\mathbb{A}_1$ - Accuracy: The ability of an AI system to provide precise and reliable outputs.

$\mathbb{A}_2$ - Efficiency: The ability of the system to reduce time and resource consumption while performing tasks effectively.

$\mathbb{A}_3$ - Safety and reliability: The degree of security, privacy protection, and operational reliability provided by the AI system.

$\mathbb{A}_4$ - User-friendliness: The ease of use and quality of interaction between users and the AI-based system.

The decision-maker evaluates the AI alternatives according to the above criteria (Table 1). The importance of the criteria is represented by the weight vector $W = (0.7897,\ 0.1302,\ 0.0288,\ 0.0514)$.
The proposed MADM approach is then applied to determine the most suitable AI-based alternative under uncertain information.

Table 1
T-SFVs decision matrix $\mathbb{D}_1$

Alt.		$\mathbb{A}_1$			$\mathbb{A}_2$			$\mathbb{A}_3$			$\mathbb{A}_4$	
$\mathbf{\Phi_1}$	0.3	0.5	0.4	0.3	0.6	0.5	0.5	0.3	0.55	0.44	0.43	0.53
$\mathbf{\Phi_2}$	0.7	0.2	0.3	0.25	0.7	0.44	0.4	0.4	0.56	0.32	0.45	0.21
$\mathbf{\Phi_3}$	0.5	0.6	0.4	0.22	0.5	0.34	0.6	0.36	0.62	0.35	0.51	0.22
$\mathbf{\Phi_4}$	0.6	0.2	0.4	0.43	0.45	0.52	0.5	0.44	0.52	0.41	0.43	0.35
$\mathbf{\Phi_5}$	0.4	0.3	0.2	0.56	0.34	0.62	0.7	0.45	0.43	0.45	0.22	0.45

Since all attributes are of the benefit type, no normalization procedure is required. The major steps involved in applying the T-SFAAA operator are described as follows.

Step 1: Using the T-SFAAA operator, aggregate each T-spherical fuzzy decision matrix $R^{\mathbb{t}} = (r_{\iota\tau}^{\mathbb{t}})_{4x5}(\mathbb{t} = 1,2,3,)$, to construct the collective T-spherical fuzzy decision matrix $R = (r_{\iota\tau})_{5\times4}$, as presented in Table 2.

Table 2
T-SFVs decision matrix R

Alt.		T-SFAAA			T-SFAAG	
$\mathbf{\Phi_1}$	0.0345	0.3381	0.1339	0.0072	0.1080	0.0676
$\mathbf{\Phi_2}$	0.2684	0.0000	0.0071	0.2363	0.1478	0.0472
$\mathbf{\Phi_3}$	0.0915	0.5571	0.0574	0.0968	0.1579	0.0648
$\mathbf{\Phi_4}$	0.1566	0.0000	0.1183	0.5022	0.0379	0.0640
$\mathbf{\Phi_5}$	0.1037	0.0055	0.0000	0.1379	0.0255	0.0964

Step 2: To obtain the overall preference values r_ι, all preference values $r_{\iota\tau}$, $(\iota = 1,2,\ldots,5)$, corresponding to the i-th row of matrix R, are aggregated using the T-SFPAAA operator (Table 3).

Table 3
Result of decision matrix

Alt.	T-SFAAA	T-SFAAG	Ranking
$\mathbf{\Phi_1}$	0.4795	0.4992	4
$\mathbf{\Phi_2}$	0.5097	0.5049	1
$\mathbf{\Phi_3}$	0.4139	0.4983	5
$\mathbf{\Phi_4}$	0.5011	0.5632	3
$\mathbf{\Phi_5}$	0.5006	0.5009	2

Step 3: Determine the ranking order of the alternatives according to the obtained score values. Since $\Phi_2 \succcurlyeq \Phi_4 \succcurlyeq \Phi_5 \succcurlyeq \Phi_1 \succcurlyeq \Phi_3$, the ranking results indicate that Φ_2 is the most desirable alternative (Figure 1).

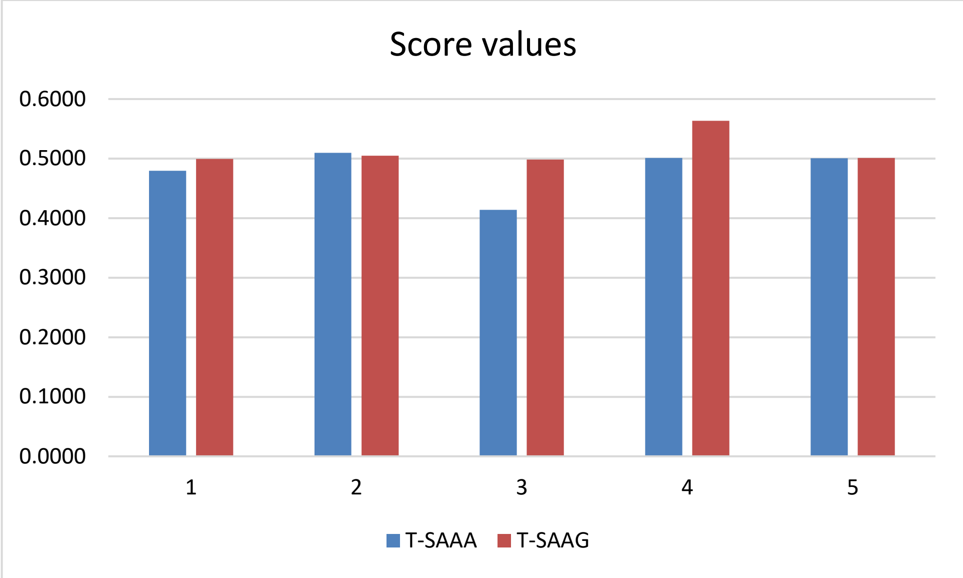

Fig. 1. Score values presented graphically

Accordingly, the ranking order of the alternatives can be expressed as $\chi_2 \succcurlyeq \chi_4 \succcurlyeq \chi_5 \succcurlyeq \chi_1 \succcurlyeq \chi_3$. Therefore, $\boldsymbol{\Phi}_2$ is identified as the optimal alternative. It is noteworthy that both the T-SFAAA and T-SFAAG operators produce identical ranking patterns, thereby confirming the stability and consistency of the proposed framework.

In this section, the results obtained using the proposed T-SFAAA and T-SFAAG operators are compared with several existing aggregation approaches for T-spherical fuzzy values, including the methods proposed by Ju *et al.,* [29], Javed *et al.,* [30], and Khan *et al.,* [27]. The comparative results for the considered decision-making problem are presented in Table 4.

Table 4
Comparative analysis

Operator	Ranking
$\boldsymbol{T-SFPAAA}$	$\chi_2 \succcurlyeq \chi_4 \succcurlyeq \chi_5 \succcurlyeq \chi_3 \succcurlyeq \chi_1$
$\boldsymbol{T-SFPAAG}$	$\chi_2 \succcurlyeq \chi_4 \succcurlyeq \chi_5 \succcurlyeq \chi_3 \succcurlyeq \chi_1$
Ju *et al.* [29]	$\chi_2 \succcurlyeq \chi_4 \succcurlyeq \chi_5 \succcurlyeq \chi_3 \succcurlyeq \chi_1$
Javed *et al.* [30]	$\chi_2 \succcurlyeq \chi_4 \succcurlyeq \chi_5 \succcurlyeq \chi_3 \succcurlyeq \chi_1$
Khan *et al.* [27]	$\chi_2 \succcurlyeq \chi_4 \succcurlyeq \chi_5 \succcurlyeq \chi_3 \succcurlyeq \chi_1$

The comparative analysis demonstrates that both the T-SFAAA and T-SFAAG operators consistently identify $\boldsymbol{\Phi}_2$ as the optimal alternative. The obtained results further confirm the effectiveness and robustness of the proposed operators. In contrast to existing approaches, the proposed framework explicitly incorporates the prioritization of attributes, which enables a more realistic and reliable representation of decision-makers' preferences under uncertain environments. Moreover, the consistency of the ranking outcomes across different aggregation operators highlights the stability and practical applicability of the proposed MADM methodology.

7. Conclusions

In this study, AHP-based Aczel–Alsina t-norm and t-conorm prioritized aggregation operators are developed for T-spherical fuzzy (T-SF) information to effectively address uncertainty in complex decision-making problems. The proposed T-SFAAA and T-SFAAG operators successfully integrate prioritization mechanisms with Aczel–Alsina operational laws within the T-spherical fuzzy environment, thereby providing greater flexibility, robustness, and reliability compared with existing aggregation approaches.

Furthermore, an AHP-based MADM framework is established and applied to artificial intelligence (AI)-driven smart systems in order to demonstrate the practical applicability of the proposed methodology. Comparative and sensitivity analyses further confirm the effectiveness, stability, and superiority of the proposed operators over several existing approaches.

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Conflicts of Interest
The authors declare no conflicts of interest.

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