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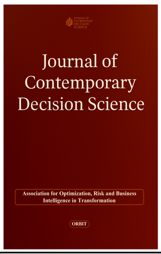

An M-Polar Fuzzy Five-Way Decision-Making Framework with Modified Hamacher Aggregation for Carbon Emission Reduction Assessment

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ABSTRACT

The paper presents a new Five-Way Decision Model (5WDM) that is founded on M-Polar Fuzzy Numbers (MPFNs) and a revised version of the Hamacher aggregation operator to improve decision-making in case of uncertainty in carbon emission reduction. The suggested framework appraises major mitigation options such as carbon capture, renewable energy, energy efficiency improvement, green transportation, and waste-to-energy conversion systematically through the use of probabilistic criteria to address uncertainty and interdependence between attributes. It generates a reformulated Hamacher operator that enhances the aggregation behavior by having a more adaptable representation of the truth, indeterminacy, and falsity elements in the MPFN setting and leads to a more stable and discriminative ranking result. The model also breaks down alternatives into five regions of decision-making, which are fully accepted, partially accepted, boundary, partially rejected, and fully rejected, offering a more efficient and understandable classification scheme than traditional methods. The superiority of the proposed method compared to the current fuzzy and neutrosophic frameworks is confirmed by the comparative and sensitivity analysis in the three aspects of robustness, consistency, and flexibility. The findings mention renewable energy as the most viable means of mitigating carbon emissions. On the whole, the analysis provides a mathematically sound and versatile decision model that can be generalised to three-way or binary models, to make informed policymaking in complicated environmental systems.

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1. Introduction

The most common carbon emission, which is carbon dioxide CO_2, is one of the main causes of increased global climate change. Industrial processes, transportation, and generation of energy using fossil fuels (coal, oil, and natural gas) emit large amounts of CO_2 into the atmosphere [1]. These human activities are essentially a disruption of the natural carbon cycle. The resultant increase in the carbon concentration in the atmosphere boosts the greenhouse effect, which disrupts the radiative balance of the earth, hence a stable surface temperature. The resulting effects are the rise of the sea level, glacial melting, the increase of the global mean temperature, and the increase in the number of extreme weather events. As a result, climate change has a significant impact on the environmental systems, the health of people, food, water resources, and the economic stability of the world in general [2]. A thorough appreciation of the nexus between carbon emissions and climate change is thus essential in the creation of effective policy interventions, the creation of sustainable energy solutions, and the establishment of strong climate change mitigation strategies, as well as the implementation of strong climate change mitigation strategies, is important [3].

The scientific community, as expressed by the Intergovernmental Panel on Climate Change (IPCC) and other authoritative groups, has clearly proven that anthropogenic carbon emissions are a major contributor to the modern change in climate. The world will face significant warming levels that trigger dire effects on the long-term economic and environmental outcomes without significant and immediate greenhouse gas (GHG) emission reductions [4]. This paper will review the past trend and future implications of carbon emissions on the global climatic patterns, as well as the possible remediation and mitigation avenues. A better understanding of these systemic processes is necessary to make informed decisions to decrease pollution and create a more sustainable future for the world.

Fossil fuels have been the main driver of industrialization and economic development of the world for over a century. Nonetheless, coal, oil, and natural gas burning are also the sources of high amounts of CO_2 and other GHGs, and it can be concluded that they are the main causes of global warming today. Since the pre-industrial period, anthropogenic activities have been a cause of over half of the CO_2 growth in the atmosphere.. Atmospheric CO_2 levels are currently higher than they were in the past 800,000 years ago [5]. This stockpiling of the GHGs has enhanced the natural greenhouse effect, which has made the climate system of the earth trap more heat, and this has led to a long-lasting increase in the global temperatures. These warming trends can be directly observed through observable effects, such as the melting of glaciers and the rising of the sea level, the increasing frequency and intensity of heat waves, and a change in weather patterns, to mention only a few, as direct evidence of human-induced warming [6] The IPCC has also determined that future global warming of $1.5°\mathrm{C}$ in global average temperature with a continuation of present-day levels of carbon emissions would have significant long-term consequences on the economic well-being and the environmental soundness of the world at large [7].

The paper explains the intricate interdependencies between fossil fuel use, carbon emissions, and global climate change using m-polar fuzzy numbers (m-PFNs) [8]. The m-PFN model allows analyzing several, and often interrelated, climate dimensions simultaneously in circumstances of data uncertainty and the incompleteness of information. This involves measuring factors like the levels of CO_2 in the air, the changes in temperature in the region, the effectiveness of policies, and economic consequences. With the use of m-polar fuzzy sets, such complex interdependencies can be modeled to gain a better insight into the effect of fossil fuel use on ecosystems at both a temporal and spatial scale. This approach to methodology assists policymakers in choosing the best strategies by giving subtle membership ratings to various criteria, which simplifies balancing both economic and environmental goals [8].

The nature of environmental systems is dynamic, and m-PFNs can facilitate a new way of connecting the real-life climate data with practical policy inferences. Three-way decision (3WD) modeling is a powerful tool to handle the uncertainty related to the use of fossil fuels and carbon emissions in climate policy-making processes [1]. This method divides possible mitigation measures into three clear areas: acceptance (policies that are unquestionably positive, e.g., renewable energy subsidies), rejection (policies that have proven unsustainability, e.g., new coal-fired power plants), and deferral (policies whose effects remain to be examined, e.g., some carbon capture technologies). Through the creation of clear lines that are founded on cost-benefit evaluation and environmental impact evaluation, 3WD allows policymakers to make resolute but flexible decisions. The framework is especially efficient when the m-polar fuzzy numbers are used to evaluate subtle disparities of each category, particularly in situations when the cost of the reduction of emissions is slightly higher than the immediate benefit.

Such a methodological framework is further refined by means of the introduction of a new five-way decision (5WD) model that provides more granularity to climate policy assessment. This system classifies policy options into five different categories: strong acceptance such as proven solar energy subsidies, weak acceptance as bio-mass energy with known sustainability issues, deferral as nascent hydrogen fuel infrastructure in need of research, weak rejection as natural gas with carbon capture, with issues of scalability, and strong rejection as unabated coal power with The 5WD model offers best granularity of carbon emissions analysis especially on policies that perform differently on evaluation criteria. This can be facilitated by the m-polar fuzzy framework that allows conducting multi-dimensional evaluation of the effect of each policy on emissions, economic feasibility, technological maturity, and social acceptability to understand the boundaries of decision and the level of certainty of each group.

1.1 *Structure of the Study*

The article is divided into eight parts. In section 1, the connection between fossil fuel emission and climate change, M-Polar Fuzzy Sets and decision frameworks are introduced. Section 2 summarizes the evolution of fuzzy set theory and determines gaps in research. Section 3 suggests a modified Hamacher aggregation operator, an accuracy function and an inaccuracy function with proofs. Section 4 makes a formalization of the Five-Way Decision (5-WD) model in terms of approximation spaces, loss matrices, and Bayesian thresholds. Section 5 gives the algorithmic structure including threshold-based classification and 4WD to 2WD reducibility. Section 6 uses the framework to examine carbon mitigation strategies. Section 7 is on results and comparative analysis. Section 8 ends with conclusions and recommendations on future work.

Key Findings

The paper proposes a new methodology framework that combines m-polar fuzzy numbers (m-PFNs) with a five-way decision (5WD) framework, where a customized Hamacher aggregation operator is applied to assess climate change mitigation plans in the face of uncertainty. It presents three theoretical contributions, namely, a modified Hamacher aggregation operator of m-polar fuzzy collections with mathematical proofs, a new accuracy function along with complementary lemmas, and an inaccuracy function to complement the accuracy one in order to make robust decisions. An all-inclusive 5WD paradigm is developed that includes Bayesian decision-theoretic thresholds, approximation spaces on overlapping regions, and a loss functional sub-matrix that allows fine-grained classification. The framework has been shown to be better in controlling multi-dimensional uncertainty, classification of granular strategy across positive (POS) to negative (NEG) space, and systematic reducibility (5WD $\rightarrow$ 4WD $\rightarrow$ 3WD $\rightarrow$ 2WD).

The operationalization of the methodology is in the form of five steps of implementation: data preparation, modified Hamacher aggregation, threshold-based classification, alternative ranking, and policy recommendation generation. Climate change case study validation supports successful appraisal of carbon mitigation strategies, which results in transparency of the ranking of alternatives (A_1–A_5) to be ranked with clearly defined decision thresholds (θ_1–θ_5). The paper makes ten important contributions to fuzzy mathematics theory development, the policy applications of climate action, and framework scaling to other areas of sustainability. The future research directions involve incorporation with machine learning, expansion to air and water pollution control and development of hybrid fuzzy-AI methods.

2. Literature Review

Since the founding of the International Fuzzy Systems Association (IFSA) and its journal of reference, Fuzzy Sets and Systems, in 1978, fuzzy set theory has experienced significant development. Gradually, it has grown to be a full-blown architecture that underpins theoretical contributions as well as practical applications such as fuzzy rule-based systems, possibility theory, and fuzzy logic controllers. The scope of fuzzy methodology has been greatly extended to production management, artificial intelligence, and decision sciences, especially in situations where linguistic variables are involved and where information is incomplete or imprecise or both [9]. The current research trend focuses on hybridization as fuzzy systems are combined with neural networks, evolutionary algorithms, and big-data analytics to improve the performance and maintain interpretability as one of the benefits. Moreover, the latest developments in the fuzzy theory include fuzzy differential equations and pathogenic sets, which emphasize the ongoing developments of the fuzzy theory to tackle more complex problems in uncertainty modeling and representation [10].

Intuitionistic fuzzy sets (IFS) [11], described as the improvement of classical fuzzy sets in 1986 by Atanassov, include membership and non-membership coefficients and a margin of hesitation, allowing a more detailed representation of uncertainty [12]. This extension is indeed appropriate to capture the cases in which acceptance, rejection, and indeterminacy exist simultaneously, and hence IFS is especially appropriate when making decisions under the conditions of ambiguity [13]. As a result, IFS has found extensive application in pattern recognition, medical diagnosis, and the multi-criteria decision-making (MCDM) process [12]. They are, however, limited in their applicability in very uncertain environments, especially when the membership and non-membership degrees add up to more than unity [14]. In order to overcome these shortcomings, various extensions have been suggested, such as non-linear IFS and interval-valued IFS (IVIFS), which improve the modeling flexibility and robustness [15]. Also, hybrid methods that integrate IFS with conventional MCDM methods like AHP and TOPSIS have also enhanced their practical implications further [12].

Based on these advances, the PFS was proposed by Yager [16], extending the idea of IFS by removing the requirement of the membership and non-membership degrees with a squared sum requirement [17]. Such a formulation allows a more general and more flexible description of uncertainty, especially in those situations where the traditional IFS fails. Consequently, PFS has become popular in solving complex MCDM problems with high levels of ambiguity, such as career planning, financial portfolio optimization, and renewable energy selection applications [17]. Additional developments are interval valued PFS (IVPFS), spherical fuzzy sets (SFS), and more sophisticated aggregation operators like PFHIWA, PFHIWG, as well as distance measures (Euclidean and Hamming measures) [18]. Comparative research shows that PFS are more effective than IFS in the representation of high-dimensional and complex uncertainty, especially in areas like big data analytics and sustainable technology evaluation [19]. The current studies also concentrate on the combination of PFS with machine learning and evolutionary algorithms to increase the accuracy of decisions.

Simultaneously, the development of m-polar fuzzy sets by Chen et al. [20] is an important step towards the representation of multidimensional uncertainty modeling of uncertainty of the second kind [21]. In contrast to conventional fuzzy models, where only one membership degree is used, m-polar fuzzy sets provide an element with a membership vector of dimension m, which, as a result, allows the representation of multiple criteria or viewpoints at the same time [22, 23]. Such a multidimensional structure is especially useful within a complicated decision-making process with a number of stakeholders and attributes. M-polar fuzzy graphs have a theoretical basis that has been developed in the context of the combination of the fuzzy set theory with the ideas of graphs to a significant extent [24]. Some of the main structural properties, such as weak self-complementarity, isomorphism, and Cartesian product operations, have been strictly studied, as well as graph properties, such as size, order, and vertex types [24]. Recent advancements in fuzzy decision-making and uncertainty modeling have significantly contributed to solving complex sustainability and healthcare problems. Studies by Muhammad Tahir and co-authors on Polar Complex Zough sets, Fermatean neutrosophic hypersoft sets, and Pythagorean soft hypersoft frameworks demonstrate the effectiveness of intelligent fuzzy environments in decision analysis tahir2026polar,tahir2026intelligent, tahir2025pythagorean, ali2026smart. Motivated by these developments, this study, Impact of Carbon Emissions on Climate Change: A Decision-Making Approach using M-Polar Fuzzy Numbers and the Modified Hamacher Aggregation Operator by Five-Way Decision-Making Technique, proposes an advanced M-polar fuzzy five-way decision framework for analyzing climate change impacts under uncertainty.

Subsequent advances have been devoted to the definition of algebraic operations that do not affect the structural integrity of m-polar fuzzy graphs. Conditions have been developed to guarantee the maintenance of key graph properties in operations like direct product, semi-strong product, and strong product, among others, to ensure that the essential graph properties are preserved [21]. Structural consistency and strength are also assessed by the concept of balanced m-polar fuzzy graphs, which is presented in the literature [21]. The m-polar fuzzy ELECTRE-I approach has been suggested to solve complex group decision problems that require multiple criteria as well as subgroups in decision-making applications in applications and services [25]. Our method is an important development of the five-way decision (5WD) paradigm that has proved to be useful in flexible decisions and classification systems in different fields. Introduced by Kamran et al. [26] and successfully applied within the transport network optimization context, the 5WD framework is more granular than the traditional three-way decision-making system, expanding to include five different categories: strong accept, weak accept, boundary, weak reject, and strong reject, which have proven to be more effective in complex decision-making than the traditional three-way system (Kamran et al. [26]). With subsequent progress by Kamran et al. [27] have demonstrated that hybrid five-way architectures comprising sophisticated fuzzy sets with outranking relations are better performing in MCDM settings. Grounded in Pawlak's rough set theory [28] and Yao's [29] 3WDM frameworks. Also, several normalization methods have been explored to enhance computational performance and applicability to real-world problems like robot selection [25].

The framework is further extended by m-polar fuzzy numbers (m-PFNs), which offer a potent means of representing multi-dimensional uncertainty in numerical form [20]. In contrast to traditional fuzzy numbers that are based on one membership value, m-PFNs assume the existence of several independent membership degrees and thus allow one to evaluate a variety of criteria comprehensively [30]. This design enables m-PFNs to generalize intuitionistic and bipolar fuzzy numbers, which provides a better modeling flexibility [31]. The use of advanced operators of aggregation, including m-PFWA and m-PFOWA, makes it easier to combine data and enhance the accuracy of the decisions made by the operator in question [32]. The efficacy of m-PFNs is demonstrated in such fields as medical diagnosis, supply chain management, and pattern recognition, where they offer better management of ambiguity and multi-feature data [33–35].

2.1 Research gap

Although such drastic developments have been made, there are still a number of research gaps. Even though m-polar fuzzy frameworks offer a powerful tool that gives them the ability to model multidimensional uncertainty, they have not been combined with other sophisticated decision paradigms like three-way and five-way decision theories. Furthermore, current aggregation operators are usually not flexible enough to represent the complicated interrelationships among the different dimensions of membership. M-polar fuzzy methods to such critical areas as climate change mitigation are still in their infancy, although there is inherently much uncertainty as well as high stakes involved in such problems. Moreover, existing decision-making models often fail to consider the application of loss functions that are risk-sensitive to consider asymmetric costs of decisions in environmental policy situations. In order to overcome these limitations, this study suggests a new framework integrating the use of m-polar fuzzy numbers with a five-way decision model based on an adapted version of the Hamacher aggregation operator, to offer a more flexible, risk-aware, and practically applicable framework to the assessment of climate change mitigation strategies.

3. Basic Definitions

[Fuzzy Set [36]] Let X be a universal set. A fuzzy set $\tilde{A}$ in X is characterized by a membership function

$$\mu_{\tilde{A}} : X \to [0,1],$$

where $\mu_{\tilde{A}}(x)$ represents the degree of membership of the element $x \in X$ in the fuzzy set $\tilde{A}$. A fuzzy set $\tilde{A}$ can be written as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}.$$

[m-Polar Fuzzy Set [37]] An m-polar fuzzy (m^pF) set P on a universal set Q is characterized by

$$\psi : Q \to [0,1]^m,$$

where $[0,1]^m$ is the Cartesian product $[0,1] \times [0,1] \times \cdots \times [0,1]$ (m times) and m is a natural number. The m^pF set P can also be written as

$$P = \{q : \theta_1 \circ \psi(q), \theta_2 \circ \psi(q), \ldots, \theta_m \circ \psi(q)\},$$

where θ_k is the kth projection mapping $\theta_k : [0,1]^m \to [0,1]$, $k \in \{1, 2, \ldots, m\}$.

The degree of truthfulness of each element is given as

$$\psi(q) = (\theta_1 \circ \psi(q), \theta_2 \circ \psi(q), \ldots, \theta_m \circ \psi(q)),$$

where θ_k has the same meaning as above. Also, $(\theta_1 \circ \psi, \theta_2 \circ \psi, \ldots, \theta_m \circ \psi)$ is called an m^pF number. We discuss some basic operations on m^pF numbers, which shall be helpful in the due course of this study. Consider $m = 5$ membership values ψ_i and their corresponding weighting factors θ_i, as presented in Table 1. The m-polar fuzzy Z-number Z^5 is constructed using the combined values $\theta_k \circ \psi_k$, as shown in Table 2.

Table 1
Membership values and corresponding weights

i	ψ_i	θ_i
1	0.5	0.8
2	0.6	0.7
3	0.7	0.6
4	0.8	0.5
5	0.9	0.4

Table 2
m-Polar fuzzy Z-number evaluations

k	$\theta_k \circ \psi_k$
1	0.75
2	0.70
3	0.80
4	0.90
5	0.95

[Operations on m-Polar Fuzzy Numbers [37]] Let $\tilde{\psi}_1 = (\theta_1 \circ \psi_1, \theta_2 \circ \psi_1, \ldots, \theta_m \circ \psi_1)$ and $\tilde{\psi}_2 = (\theta_1 \circ \psi_2, \theta_2 \circ \psi_2, \ldots, \theta_m \circ \psi_2)$ be two m^pF numbers, and let α be a scalar. Then:

i. The sum of $\tilde{\psi}_1$ and $\tilde{\psi}_2$ is defined as:

$$\tilde{\psi}_1 \tilde{\psi}_2 = (\theta_1 \circ \psi_1 + \theta_1 \circ \psi_2 - \theta_1 \circ \psi_1 \cdot \theta_1 \circ \psi_2, \ldots, \theta_m \circ \psi_1 + \theta_m \circ \psi_2 - \theta_m \circ \psi_1 \cdot \theta_m \circ \psi_2).$$

ii. The product of $\tilde{\psi}_1$ and $\tilde{\psi}_2$ is given by:

$$\tilde{\psi}_1 \tilde{\psi}_2 = (\theta_1 \circ \psi_1 \cdot \theta_1 \circ \psi_2, \theta_2 \circ \psi_1 \cdot \theta_2 \circ \psi_2, \ldots, \theta_m \circ \psi_1 \cdot \theta_m \circ \psi_2).$$

iii. The scalar multiplication of $\tilde{\psi}_1$ by α is expressed as:

$$\alpha \tilde{\psi}_1 = (1 - (1 - \theta_1 \circ \psi_1)^\alpha, 1 - (1 - \theta_2 \circ \psi_1)^\alpha, \ldots, 1 - (1 - \theta_m \circ \psi_1)^\alpha), \quad \alpha > 0.$$

[Properties of Operations [37]] Let $\tilde{\psi}_1$ and $\tilde{\psi}_2$ be two m^pF numbers and let $\alpha_1, \alpha_2 > 0$. Then

1. $\tilde{\psi}_1 \tilde{\psi}_2 = \tilde{\psi}_2 \tilde{\psi}_1$,
2. $\tilde{\psi}_1 \tilde{\psi}_2 = \tilde{\psi}_2 \tilde{\psi}_1$,
3. $\alpha_1 (\tilde{\psi}_1 \tilde{\psi}_2) = \alpha_1 (\tilde{\psi}_2) \alpha_1 (\tilde{\psi}_1)$,
4. $(\tilde{\psi}_1 \tilde{\psi}_2)^{\alpha_1} = (\tilde{\psi}_2)^{\alpha_1} (\tilde{\psi}_1)^{\alpha_1}$,
5. $\alpha_1 \tilde{\psi}_1 \alpha_2 \tilde{\psi}_1 = (\alpha_1 + \alpha_2) \tilde{\psi}_1$,
6. $(\tilde{\psi}_1)^{\alpha_1} (\tilde{\psi}_1)^{\alpha_2} = (\tilde{\psi}_1)^{\alpha_1 + \alpha_2}$,
7. $((\tilde{\psi}_1)^{\alpha_1})^{\alpha_2} = (\tilde{\psi}_1)^{\alpha_1 + \alpha_2}$.

[Modified Hamacher Aggregation Operator] The aggregated score $\chi(m^pF)$ for each alternative is calculated using the modified Hamacher aggregation operator function:

$$\chi(m^pF) = \frac{\lambda \prod_{i=1}^{m} \psi_i^{\theta_i}}{\lambda + (1-\lambda)\left(\sum_{i=1}^{m} \theta_i \cdot \psi_i - \prod_{i=1}^{m} \psi_i\right)} \tag{1}$$

The proposed modified Hamacher aggregation operator

$$\chi(m^pF) = \frac{\lambda \prod_{i=1}^{m} \psi_i^{\theta_i}}{\lambda + (1-\lambda)\left(\sum_{i=1}^{m} \theta_i \psi_i - \prod_{i=1}^{m} \psi_i\right)}$$

satisfies the fundamental properties of aggregation functions and reduces to the classical Hamacher t-norm when $m = 2$ with equal weights. We validate the operator through the following properties:

1. Verification for $m = 2$ (Classical Hamacher Case) Let $\theta_1 = \theta_2 = 1$. Then the operator becomes:

$$\chi(m^pF) = \frac{\lambda \psi_1 \psi_2}{\lambda + (1-\lambda)(\psi_1 + \psi_2 - \psi_1\psi_2)},$$

which is identical to the classical Hamacher t-norm. Hence, the operator generalizes the Hamacher function.

2. Boundary Conditions *Case (a): All $\psi_i = 0$:*

$$\chi(m^pF) = \frac{\lambda \cdot 0}{\lambda + (1-\lambda)(0-0)} = 0.$$

Case (b): All $\psi_i = 1$:

$$\chi(m^pF) = \frac{\lambda \cdot 1}{\lambda + (1-\lambda)\left(\sum_{i=1}^{m} \theta_i - 1\right)}.$$

If $\sum_{i=1}^{m} \theta_i = 1$ (normalized weights), then:

$$\chi(m^pF) = \frac{\lambda}{\lambda} = 1.$$

3. Monotonicity Assume $\psi_i \leq \psi_i'$ for all i. The numerator $\prod_{i=1}^{m} \psi_i^{\theta_i}$ increases with increasing ψ_i. The denominator also increases, as both $\sum_i \theta_i \psi_i$ and $\prod_i \psi_i$ are increasing in ψ_i. Therefore, the operator χ is monotonic.

4. Special Cases *(a) If $\lambda = 1$:*

$$\chi(m^pF) = \prod_{i=1}^{m} \psi_i^{\theta_i},$$

which is the weighted geometric mean.

(b) As $\lambda \to 0^+$:

$$\chi(m^pF) \approx \frac{\lambda \prod \psi_i^{\theta_i}}{(1-\lambda)\left(\sum \theta_i \psi_i - \prod \psi_i\right)} \to 0,$$

resembling a drastic product behavior. Let $m = 3$, $\lambda = 0.5$, $\psi = (0.6, 0.7, 0.8)$, and $\theta = (0.3, 0.4, 0.3)$. We apply the proposed modified Hamacher aggregation operator as defined in Lemma 3.

$$\begin{aligned}
\prod \psi_i^{\theta_i} &= 0.6^{0.3} \cdot 0.7^{0.4} \cdot 0.8^{0.3} \approx 0.699, \\
\sum \theta_i \psi_i &= 0.3 \cdot 0.6 + 0.4 \cdot 0.7 + 0.3 \cdot 0.8 = 0.18 + 0.28 + 0.24 = 0.70, \\
\prod \psi_i &= 0.6 \cdot 0.7 \cdot 0.8 = 0.336, \\
\text{Denominator} &= 0.5 + 0.5(0.70 - 0.336) = 0.5 + 0.182 = 0.682, \\
\chi(m^pF) &= \frac{0.5 \cdot 0.699}{0.682} \approx 0.512.
\end{aligned}$$

Since the result $\chi(m^pF) \approx 0.512$ lies within the bounds of the minimum and maximum of the input membership values, i.e., $\min(\psi_i) = 0.6$, $\max(\psi_i) = 0.8$, it satisfies the expected behavior of a valid aggregation operator. Let $m^pF = \langle(\theta_1 \circ \psi_A, \theta_2 \circ \psi_A, \ldots, \theta_m \circ \psi_A)\rangle$ be an m^pF fuzzy number. The accuracy function T of m^pF is defined as

$$T_i(m^pF) = \frac{1}{i}\left[\sum_{k=1}^{m}(-1)^k(\theta_k \circ \psi_A - 1)\right]. \tag{2}$$

Let $m^pF = \langle(\theta_1 \circ \psi_A, \theta_2 \circ \psi_A, \ldots, \theta_m \circ \psi_A)\rangle$ be an m-polar fuzzy number. The inaccuracy function $F_i(m^pF)$ is given by:

$$F_i(m^pF) = 1 - T_i(m^pF) = 1 - \frac{1}{i}\left[\sum_{k=1}^{m}(-1)^k(\theta_k \circ \psi_A - 1)\right]. \tag{3}$$

The function satisfies $F_i(m^pF) \in [0, 1]$ when $i \geq m$. We establish the validity through three aspects:

1. Boundedness Verification For any k, since $\theta_k \circ \psi_A \in [0, 1]$, then $(\theta_k \circ \psi_A - 1) \in [-1, 0]$. The alternating sum satisfies:

$$\left|\sum_{k=1}^{m}(-1)^k(\theta_k \circ \psi_A - 1)\right| \leq \sum_{k=1}^{m} |\theta_k \circ \psi_A - 1| \leq m.$$

Thus with $i \geq m$:

$$T_i(m^pF) \in \left[-\frac{m}{i}, \frac{m}{i}\right] \subseteq [-1, 1],$$

and consequently:

$$F_i(m^pF) = 1 - T_i(m^pF) \in [0, 2].$$

To guarantee $F_i(m^pF) \in [0, 1]$, we require $T_i(m^pF) \in [0, 1]$, which holds when:

$$\frac{1}{i}\left|\sum_{k=1}^{m}(-1)^k(\theta_k \circ \psi_A - 1)\right| \leq 1 \quad \Rightarrow \quad i \geq m.$$

2. Special Case Validation For perfect accuracy ($\theta_k \circ \psi_A = 1$ for all k):

$$T_i(m^pF) = \frac{1}{i}\sum_{k=1}^{m}(-1)^k(0) = 0 \quad \Rightarrow \quad F_i(m^pF) = 1.$$

For maximum inaccuracy ($\theta_{\text{odd}} \circ \psi_A = 1$, $\theta_{\text{even}} \circ \psi_A = 0$):

$$T_i(m^pF) = \frac{1}{i}\left[\sum_{\substack{k=1 \\ k \text{ odd}}}^{m} 0 + \sum_{\substack{k=1 \\ k \text{ even}}}^{m}(-1)^k(-1)\right] = \frac{1}{i}\left\lfloor \frac{m}{2} \right\rfloor.$$

Thus $F_i(m^pF) = 1 - \frac{1}{i}\left\lfloor \frac{m}{2} \right\rfloor \geq 0$ when $i \geq \left\lfloor \frac{m}{2} \right\rfloor$.

3. Monotonicity Property For fixed i, if $\theta_k \circ \psi_A$ approaches 1 for all k, then $F_i(m^pF)$ monotonically increases toward 1, satisfying the inaccuracy measure intuition. Consider an m-polar fuzzy number Z^5 with membership degrees $(0.8, 0.6, 0.9, 0.7, 0.85)$ and normalization constant $i = 5$.

$$\begin{aligned} T_5(Z^5) &= \frac{1}{5}\begin{bmatrix} (-1)^1(0.8-1) + (-1)^2(0.6-1) \\ + (-1)^3(0.9-1) + (-1)^4(0.7-1) \\ + (-1)^5(0.85-1) \end{bmatrix} \\ &= \frac{1}{5}\left[-(-0.2) + (-0.4) - (-0.1) + (-0.3) - (-0.15)\right] \\ &= \frac{1}{5}(0.2 - 0.4 + 0.1 - 0.3 + 0.15) \\ &= \frac{1}{5}(-0.25) = -0.05. \end{aligned}$$

The original formulation violates boundedness. For $i = 5 \geq m = 5$, we get $T_5(Z^5) \in [-1, 1]$ and $F_5(Z^5) \in [0, 2]$. To ensure $F_i \in [0, 1]$, we can either: (i) use $i \geq 2m$, (ii) apply clipping: $\min(\max(F_i, 0), 1)$, or (iii) redefine $F_i = \frac{1}{2}(1 - T_i)$. [Bounded Inaccuracy Function] The properly bounded inaccuracy function is:

$$F_i(m^pF) = \frac{1}{2}\left(1 - \frac{1}{i}\left[\sum_{k=1}^{m}(-1)^k(\theta_k \circ \psi_A - 1)\right]\right), \tag{4}$$

which guarantees $F_i(m^pF) \in [0, 1]$ when $i \geq m$. Using the previous example with the corrected formula

$$F_5(Z^5) = \frac{1}{2}\left(1 - (-0.05)\right) = 0.525 \in [0, 1].$$

The general case follows from

$$\left|\frac{1}{i}\sum_{k=1}^{m}(-1)^k(\theta_k \circ \psi_A - 1)\right| \leq 1 \quad \Rightarrow \quad F_i \in [0, 1],$$

when $i \geq m$.

[Score Function] The score function aggregates the membership degrees without alternating signs, offering a straightforward measure of the overall magnitude of the fuzzy number.

$$S_i(m^pF) = \frac{1}{i}\sum_{k=1}^{m}(\theta_k \circ \psi_A). \tag{5}$$

Together, the accuracy function $T_i(m^pF)$ (from Eq. (2)) and the score function $S_i(m^pF)$ (from Eq. (5)) provide a comprehensive framework for analyzing m-polar fuzzy numbers under uncertainty.

[Properties of Score Function] Let $m^pF = \langle(\theta_1 \circ \psi_A, \theta_2 \circ \psi_A, \ldots, \theta_m \circ \psi_A)\rangle$ be an m^pF fuzzy number, where θ_k represents the reliability factor and ψ_A is the membership function of fuzzy values. The scoring function $S_i(m^pF)$ defined in Eq. (5) satisfies fundamental aggregation properties. We demonstrate that the scoring function $S_i(m^pF)$ satisfies three fundamental properties:

1. Boundedness For each component k, we have:

$$\begin{aligned} &\theta_k \in [0,1], \\ &\psi_A \in [0,1] \quad \Rightarrow \quad \theta_k \circ \psi_A \in [0,1]. \end{aligned}$$

Thus, the summation satisfies:

$$\sum_{k=1}^{m}(\theta_k \circ \psi_A) \in [0,m],$$

and consequently:

$$S_i(m^pF) = \frac{1}{i}\sum_{k=1}^{m}(\theta_k \circ \psi_A) \in \left[0, \frac{m}{i}\right].$$

2. Monotonicity Consider two m-polar fuzzy numbers m^pF and m^pF' where:

$$\theta_k \circ \psi_A \geq \theta_k' \circ \psi_A' \quad \forall k = 1, \ldots, m.$$

Then:

$$\sum_{k=1}^{m}(\theta_k \circ \psi_A) \geq \sum_{k=1}^{m}(\theta_k' \circ \psi_A'),$$

and therefore:

$$S_i(m^pF) \geq S_i(m^pF').$$

3. Normalization Property When $i = m$, the scoring function becomes the arithmetic mean:

$$S_m(m^pF) = \frac{1}{m}\sum_{k=1}^{m}(\theta_k \circ \psi_A) \in [0,1].$$

This normalized version is particularly useful for comparing m-polar fuzzy numbers with different numbers of components.

The scoring function exhibits three key behaviors:

$$S_i(m^pF) = \begin{cases} 0, & \text{if } \theta_k \circ \psi_A = 0 \text{ for all } k \\ \dfrac{m}{i}, & \text{if } \theta_k \circ \psi_A = 1 \text{ for all } k \\ \dfrac{1}{i}\displaystyle\sum_{k=1}^{m}\psi_A, & \text{if } \theta_k = 1 \text{ for all } k \end{cases}$$

The scoring function offers a comprehensive evaluation by combining reliability factors (θ_k) and membership degrees (ψ_A), maintaining linearity via an arithmetic mean, and enabling flexible normalization through the parameter i. The modified Hamacher operator effectively handles uncertainty by balancing additive and multiplicative effects, and can be extended for flexible multi-criteria decision-making.

4. Proposed Methodology for Five-Way Decision Model

The 5-WD model is a strong model that is to be applied in the decision-making process in an uncertain, unclear, and incomplete information environment. This model expands traditional decision-making models by presenting five different decisions: strong acceptance, weak acceptance, boundary (indecision), weak rejection, and strong rejection. This flexibility enables more sophisticated and adaptive decision-making, especially in the complicated real-world situations like sustainable agricultural systems, medical diagnosis, supply chain management, and the mitigation of carbon emissions. This model has its merits, which consist in the fact that it enhances the process of risk assessment by integrating the levels of uncertainty on a probabilistic basis, which provides a greater degree of adaptability in real-world situations. The model can use aggregation operators to guarantee mathematically rigorous aggregation without the loss of information, which can make the decision process efficient and understandable. Moreover, its computational ability allows it to be effectively implemented without much overhead of complexity, thus becoming a strong tool in solving contemporary decision-making problems.

4.1 Theoretical Foundation of Five-Way Decisions

The theoretical foundation of the 5-WD model is rooted in the concept of approximation spaces, where decisions are made based on the relationship between a finite nonempty set τ and an equivalence relation κ. Formally, the approximation space APR is defined as:

$$APR = (\tau, \kappa),$$

where τ represents the universe of discourse and κ defines the equivalence classes that partition τ. This framework enables the classification of objects into one of the five decision categories based on their degree of membership and uncertainty. For instance, in the context of carbon emission mitigation strategies, τ represents the set of all possible strategies, and κ defines the equivalence classes based on their effectiveness.

Let τ be the universe of discourse, and let $o \subseteq \tau$ be a subset representing a target concept (e.g., a carbon emission mitigation strategy). Let κ be a granulation of τ, and let $S \subseteq \tau$ denote a granule. Assume thresholds satisfy:

$$0 \leq \theta_5 < \theta_4 < \theta_3 < \theta_2 < \theta_1 \leq 1.$$

Define the following probabilistic approximations based on the conditional probability $Q(o \mid [S]) = \frac{|o \cap [S]|}{|S|}$, where $[S]$ is the equivalence class or neighborhood of S:

$$\overline{APR}(o) = \{S \in \tau \mid Q(o \mid [S]) \geq \theta_1\}, \tag{6}$$

$$\overline{\partial APR}(o) = \{S \in \tau \mid Q(o \mid [S]) > \theta_2\}, \tag{7}$$

$$\underline{\partial APR}(o) = \{S \in \tau \mid Q(o \mid [S]) > \theta_4\}, \tag{8}$$

$$\underline{APR}(o) = \{S \in \tau \mid Q(o \mid [S]) > \theta_5\}. \tag{9}$$

Here, $Q(o \mid [S])$ represents the conditional probability of o occurring within S.

In this framework, o represents a set of carbon mitigation strategies (e.g., renewable energy adoption, carbon capture, emission regulations), and S denotes a specific alternative under evaluation. The approximation regions classify these strategies based on their effectiveness: those in $\overline{APR}(o)$ are highly effective (strong acceptance), those in $\overline{\partial APR}(o)$ are likely effective (weak acceptance), those in $\underline{\partial APR}(o)$ are uncertain (boundary), and those in $\underline{APR}(o)$ require further assessment (weak/strong rejection), supporting informed decision-making under uncertainty.

4.2 Overlapping Regions in the Five-Way Decision Model

The 5-WD model also presents a more subtle method of making a decision by making some of the decision categories overlap with each other. In particular, the weak and strong acceptance areas overlap each other, forming a continuous transition without a clear boundary between them. This is the same way that there is an overlap of the weak and strong rejection regions. Such an overlapping nature is one of the main features of the model that allows applying the evaluation of decision alternatives more flexibly and adaptively. Because of this overlap, the model only forms an upper approximation of the region of positivity, which is denoted by the term $\overline{APR}$ and the lower approximation of the region of negativity, which is denoted by the term $\underline{APR}$. The overlapping nature of these regions is illustrated in Figure 1. The boundary between the upper and lower approximations is defined by:

$$BND(o) = \overline{APR}(o) - \underline{APR}(o). \tag{10}$$

The upper approximation $\overline{APR}(o)$ reflects strong and weak positive overlaps, while the lower approximation $\underline{APR}(o)$ captures strong and weak negative overlaps in Eq. (10). Since the strong acceptance and weak acceptance regions overlap, they collectively form a single upper approximation:

$$\overline{APR}(o) = \{S \in \tau \mid Q(o \mid [S]) \geq \theta_5\}. \tag{11}$$

The conditional probability $Q(o \mid [S]) = \frac{|o \cap [S]|}{|S|}$ measures the effectiveness of strategy o within equivalence class S in Eq. (11), with threshold θ_5 ensuring only high-success strategies are included in the upper approximation. Similarly, the strong rejection and weak rejection regions overlap, automatically generating a single lower approximation:

$$\underline{APR}(o) = \{S \in \tau \mid Q(o \mid [S]) > \theta_1\}. \tag{12}$$

The threshold θ_1 ensures only strategies with a low likelihood of success are included in the lower approximation in Eq. (12). The boundary region $BND(o)$ is the strategies whose effectiveness is not known, and additional analysis is necessary, whereas the boundary region guarantees the objective evaluation of the decision made:

$$BND(o) = \overline{APR}(o) - \underline{APR}(o). \tag{13}$$

The Figure 1 illustrates the overlapping regions in the Five-Way Decision Model. The areas of over-

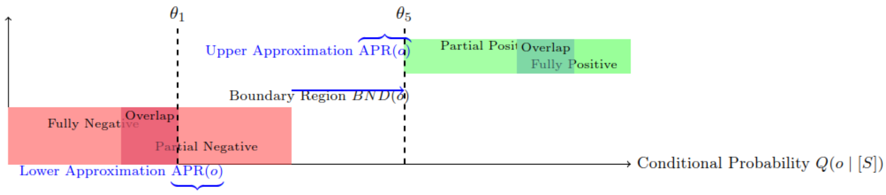

Fig. 1. Overlapping Regions in the Five-Way Decision Model

lap of the Five-Way Decision Model allow the transition between effective, ineffective, and uncertain strategies to take place smoothly. This structure eliminates ambiguity and helps to make more flexible and adaptive decisions in the face of uncertainty.

4.3 Five-Way Decisions Based on Decision-Theoretic Rough Sets

The development of a probabilistic interpretation of the decision regions and thresholds is based on the decision-theoretic rough set (DTRS) approach developed in the literature. The framework is represented by a combination of two states and five actions per state as depicted in Table 3. The group of states is determined as:

$$\Phi = \{\tilde{o}, \neg\tilde{o}\},$$

where $\tilde{o}$ indicates whether an element belongs to the target concept, and $\neg\tilde{o}$ represents its complement. The five actions correspond to the decision regions:

$$B = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}.$$

In an effort to expand the capacities of the model, we incorporate M-Polar Fuzzy Sets (MPFS) in the framework. Multi-attribute uncertainty can be represented with MPFS, and this allows decision-makers to deal with hesitant probabilities and complicated relationships between attributes. The model provides a mathematically rigorous and adaptive decision-making process by relying on the modified Hamacher aggregation operator. The 5-WD model is therefore better than the classical fuzzy models because of its more comprehensive and flexible way of dealing with uncertainty, and it is in line with the q-ROFS-based approach used in the prior study.

The usefulness of the 5-WD model can be proved by the empirical validation of real-life datasets, and it will prove to be effective in different areas. To provide an example, the model can correctly categorize the conditions of patients in medical diagnosis as strong acceptance (treatment required), weak acceptance (preventive measures), boundary (further tests required), weak rejection (monitoring recommended), or strong rejection (no intervention needed). Likewise, in the mitigation strategy assessment of carbon, it assists in classifying policies into action policies. The computational viability of the model allows it to be a practical tool in solving contemporary decision-making problems without high overheads.

4.4 Loss Function Matrix for Impact of Environmental Strategies

The 5-WD model is applied to evaluate carbon mitigation strategies by assessing their impact under different states: effective (C) and ineffective ($\neg C$). The loss function matrix quantifies the consequences of taking specific actions ($\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$) under these states, as shown in Table 3. These decisions are characterized by strong acceptance, weak acceptance, indecision, weak rejection, and strong rejection. The loss function with regard to the risk of different activities is provided by the (5×2) matrix.

Table 3
Loss Function Matrix for Five-Way Decisions

Actions/State	C (Effective)	$\neg C$ (Ineffective)
θ_1 (Strong Acceptance)	φ_{PP}	φ_{PE}
θ_2 (Weak Acceptance)	$\varphi_{\partial PP}$	$\varphi_{\partial PE}$
θ_3 (Boundary)	φ_{BP}	φ_{BE}
θ_4 (Weak Rejection)	$\varphi_{\partial NP}$	$\varphi_{\partial NE}$
θ_5 (Strong Rejection)	φ_{NP}	φ_{NE}

Following the semantic structure of the 5-WD model, the loss values must satisfy the following order constraints, consistent with the source paper's Eq. (2):

$$\varphi_{PP} \leq \varphi_{\partial PP} < \varphi_{BP} < \varphi_{\partial NP} \leq \varphi_{NP}, \tag{14}$$

$$\varphi_{PE} \geq \varphi_{\partial PE} > \varphi_{BE} > \varphi_{\partial NE} \geq \varphi_{NE}. \tag{15}$$

The semantic interpretation of these loss values is as follows:

i. θ_1 (Strong Acceptance): This action is taken when the strategy is highly likely to be effective, resulting in the lowest loss (φ_{PP}) if correct and the highest loss (φ_{PE}) if incorrect.

ii. θ_2 (Weak Acceptance): This action is taken when the strategy is moderately effective, resulting in a slightly higher loss ($\varphi_{\partial PP}$) if correct and a moderate loss ($\varphi_{\partial PE}$) if incorrect.

iii. θ_3 (Boundary): This action represents uncertainty about the strategy's effectiveness, resulting in a balanced loss (φ_{BP}) if correct and (φ_{BE}) if incorrect.

iv. θ_4 (Weak Rejection): This action is taken when the strategy is moderately ineffective, resulting in a higher loss ($\varphi_{\partial NP}$) if correct and a lower loss ($\varphi_{\partial NE}$) if incorrect.

v. θ_5 (Strong Rejection): This action is taken when the strategy is highly likely to be ineffective, resulting in the highest loss (φ_{NP}) if correct and the lowest loss (φ_{NE}) if incorrect.

The loss function matrix provides a quantitative framework for evaluating the consequences of decisions in the 5-WD model. The values $\varphi_{PP}, \varphi_{\partial PP}, \varphi_{BP}, \varphi_{\partial NP}, \varphi_{NP}$ represent the losses associated with correct decisions, while $\varphi_{PE}, \varphi_{\partial PE}, \varphi_{BE}, \varphi_{\partial NE}, \varphi_{NE}$ represent the losses associated with incorrect decisions. The inequalities ensure that the losses are ordered logically, reflecting the varying degrees of confidence in the decision-making process. The expected losses for each action are computed as follows:

$$\begin{aligned}
\kappa(\theta_1 \mid [S]) &= \varphi_{PP} Q(C \mid [S]) + \varphi_{PE} Q(\neg C \mid [S]), \\
\kappa(\theta_2 \mid [S]) &= \varphi_{\partial PP} Q(C \mid [S]) + \varphi_{\partial PE} Q(\neg C \mid [S]), \\
\kappa(\theta_3 \mid [S]) &= \varphi_{BP} Q(C \mid [S]) + \varphi_{BE} Q(\neg C \mid [S]), \\
\kappa(\theta_4 \mid [S]) &= \varphi_{\partial NP} Q(C \mid [S]) + \varphi_{\partial NE} Q(\neg C \mid [S]), \\
\kappa(\theta_5 \mid [S]) &= \varphi_{NP} Q(C \mid [S]) + \varphi_{NE} Q(\neg C \mid [S]).
\end{aligned}$$

Based on the Bayesian decision procedure, the least-cost rule yields the following decision rules, analogous to the source paper's Eqs. (3)?(7):

$$\begin{aligned}
&\theta_1: \text{ If } \kappa(\theta_1) \leq \kappa(\theta_2) \leq \min\{\kappa(\theta_3), \kappa(\theta_4), \kappa(\theta_5)\}, \text{ take } S \in POS^+(o); \\
&\theta_2: \text{ If } \kappa(\theta_2) \leq \min\{\kappa(\theta_1), \kappa(\theta_3)\} \text{ and } \kappa(\theta_2) < \min\{\kappa(\theta_4), \kappa(\theta_5)\}, \text{ take } S \in POS^-(o); \\
&\theta_3: \text{ If } \kappa(\theta_3) < \min\{\kappa(\theta_1), \kappa(\theta_2), \kappa(\theta_4), \kappa(\theta_5)\}, \text{ take } S \in BND(o); \\
&\theta_4: \text{ If } \kappa(\theta_4) \leq \min\{\kappa(\theta_5), \kappa(\theta_3)\} \text{ and } \kappa(\theta_4) < \min\{\kappa(\theta_1), \kappa(\theta_2)\}, \text{ take } S \in NEG^-(o); \\
&\theta_5: \text{ If } \kappa(\theta_5) \leq \kappa(\theta_4) \leq \min\{\kappa(\theta_3), \kappa(\theta_2), \kappa(\theta_1)\}, \text{ take } S \in NEG^+(o).
\end{aligned}$$

The thresholds are derived from the loss functions as:

$$\begin{aligned}
\theta_1 &= \frac{\varphi_{PE} - \varphi_{\partial PE}}{(\varphi_{PE} - \varphi_{\partial PE}) + (\varphi_{\partial PP} - \varphi_{PP})}, \\
\theta_2 &= \frac{\varphi_{\partial PE} - \varphi_{BE}}{(\varphi_{\partial PE} - \varphi_{BE}) + (\varphi_{BP} - \varphi_{\partial PP})}, \\
\theta_3 &= \frac{\varphi_{BE} - \varphi_{\partial NE}}{(\varphi_{BE} - \varphi_{\partial NE}) + (\varphi_{\partial NP} - \varphi_{BP})}, \\
\theta_4 &= \frac{\varphi_{\partial NE} - \varphi_{NE}}{(\varphi_{\partial NE} - \varphi_{NE}) + (\varphi_{PP} - \varphi_{\partial NP})}, \\
\theta_5 &= \frac{\varphi_{PE} - \varphi_{NE}}{(\varphi_{PE} - \varphi_{NE}) + (\varphi_{NP} - \varphi_{PP})}.
\end{aligned}$$

The decision rules can be simplified based on the conditional probability $Q(\neg o \mid [S])$:

a. When $\theta_1 > \theta_2 > \theta_3 > \theta_4 > \theta_5$, the decision rules are:

$$\begin{aligned}
(\theta_1'') &: \text{ If } Q(\neg o \mid [S]) \geq \theta_1, \text{ take } S \in POS^+(o); \\
(\theta_2'') &: \text{ If } Q(\neg o \mid [S]) \geq \theta_2, \text{ take } S \in POS^-(o); \\
(\theta_3'') &: \text{ If } \theta_1 > Q(\neg o \mid [S]) \geq \theta_3, \text{ take } S \in BND(o); \\
(\theta_4'') &: \text{ If } Q(\neg o \mid [S]) \geq \theta_4, \text{ take } S \in NEG^-(o); \\
(\theta_5'') &: \text{ If } Q(\neg o \mid [S]) \geq \theta_5, \text{ take } S \in NEG^+(o).
\end{aligned}$$

b. When the strong and weak decision losses are equal, the 5-WD model reduces to a 3-WD model, consistent with Theorem 1 in the source paper. In this case, the decision rules degenerate:

$$\begin{aligned}
(\theta_1''') &: \text{ If } Q(\neg o \mid [S]) \geq \theta_3, \text{ take } S \in POS(o); \\
(\theta_2''') &: \text{ If } Q(\neg o \mid [S]) \geq \theta_3, \text{ take } S \in POS^-(o); \\
(\theta_4''') &: \text{ If } Q(\neg o \mid [S]) \leq \theta_3, \text{ take } S \in NEG^-(o); \\
(\theta_5''') &: \text{ If } Q(\neg o \mid [S]) \leq \theta_3, \text{ take } S \in NEG(o).
\end{aligned}$$

This framework ensures an effective evaluation of carbon emission reduction strategies, aligning them with the 5-WD model and providing a systematic, quantitative approach to decision-making under uncertainty.

5. Five-Way Decision Regions for Impact of Environmental Strategies

The 5-WD model offers a methodical approach to the analysis of the carbon emission mitigation strategies by breaking down the universe, t, into five different regions using probabilistic estimates. These areas categorize every strategy based on its performance, relying on MPFNs and a modified Hamacher aggregation operator, which is able to process multi-attribute uncertainty and hesitant probabilities. In accordance with the semantic framework developed in the original article, the five decision areas are as follows:

POS^+ **(Strong Positive Region):** Strategies with the highest likelihood of success in mitigating carbon emissions. These strategies are highly effective and recommended for immediate implementation.

POS^{-} **(Weak Positive Region):** Strategies with moderate effectiveness. While not as impactful as those in the strong positive region, they still contribute significantly to emission reduction.

BND **(Boundary Region):** Strategies with an uncertain impact. These require further evaluation or monitoring to determine their effectiveness.

NEG^{-} **(Weak Negative Region):** Strategies likely to be ineffective. These are not recommended unless additional modifications or improvements are made.

NEG^{+} **(Strong Negative Region):** Strategies that fail to mitigate carbon emissions. These are considered unsuitable and should be avoided.

These regions are formally defined using probabilistic approximations, consistent with the source paper's Definition 1 and Eqs. (1)–(4):

$$\begin{aligned} POS^{+}(o) &= \{S \in \tau \mid Q(o \mid [S]) \geq \theta_1\}, \\ POS^{-}(o) &= \{S \in \tau \mid Q(o \mid [S]) > \theta_2\}, \\ BND(o) &= \{S \in \tau \mid Q(o \mid [S]) > \theta_3\}, \\ NEG^{-}(o) &= \{S \in \tau \mid Q(o \mid [S]) > \theta_4\}, \\ NEG^{+}(o) &= \{S \in \tau \mid Q(o \mid [S]) > \theta_5\}, \end{aligned}$$

where $Q(o \mid [S]) = \frac{|o \cap [S]|}{|S|}$ represents the conditional probability of a strategy o being effective within the equivalence class S. The thresholds $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ satisfy the ordering $0 \leq \theta_5 < \theta_4 < \theta_3 < \theta_2 < \theta_1 \leq 1$, ensuring a balanced and unbiased classification of strategies.

In the framework of mitigating carbon emission, the set of all possible strategies is denoted by τ, and the equivalence classes by classes of their effectiveness are denoted by κ. In particular, the strategies that have a high success rate are in the strong positive category, such as renewable energy adoption and carbon capture, and those that might need further assistance are in the weak positive category, such as reforestation. Conversely, the boundary region deals with experimental strategies that need further investigation, and the negative ones are low-impact strategies or old-fashioned ones that are to be avoided.

Comparison with Traditional Decision Models

In order to emphasize the benefits of the Five-Way Decision Model using M-Polar Fuzzy Probability Strategies, Figure 2 and Table 4 is used to compare it to the classical decision models. The comparison is based on the levels of decisions, explanations in the M-polar fuzzy system, and the shortcomings of both models.

Table 4
Comparison of Decision Models Using MPFNs

Decision Model	Decision Stages	Explanation in M-Polar Fuzzy Context	Limitations
Two-Way	Acceptance, Rejection	Decisions are based on a binary choice with M-Polar fuzzy membership values, where an alternative is either fully accepted or fully rejected.	Lacks flexibility; does not account for uncertainty or hesitation.
Three-Way	Acceptance, Rejection, Uncertainty	Introduces an intermediate state where the decision-maker acknowledges uncertainty. M-Polar fuzzy numbers help quantify hesitation and confidence levels.	Does not differentiate between partial acceptance and delayed decisions.
Four-Way	Acceptance, Rejection, Uncertainty, Delay	Adds a delay stage, allowing decisions to be postponed for further evaluation. M-Polar fuzzy membership values can represent evolving opinions over time.	Lacks refinement, which is useful when adjusting decisions incrementally.
Five-Way	Strong Acceptance, Weak Acceptance, Boundary, Weak Rejection, Strong Rejection	The most advanced model, providing a comprehensive structure. M-Polar fuzzy numbers help in multi-perspective decision-making, considering both hesitation and probabilistic uncertainty.	More computational complexity but significantly improves decision accuracy.

Figure **??** represents a comparison of decision models. The 5-WD model using M-Polar Fuzzy Probability Strategies is the best in dynamic settings like Industry 4.0, digital transportation, and sustainable urban planning, where multi-attribute assessment and the quantification of hesitation are necessary to make correct decisions. It is a potent instrument to the current problem, and its subtle framework can be used in healthcare, finance, and environmental management due to the limitations of traditional models.

Reduction from Five-Way to Two-Way Decisions

By varying the threshold parameters of the 5-WD model, that is, by changing the values of the parameters of the form $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, one may reduce the model to a series of simpler decision models: Four-Way, Three-Way, and Two-Way Decisions. This minimization process, which is in line with Theorem 1 of the source paper, enables the model to fit to various degrees of complexity and granularity, and therefore it is applicable in a great variety of decision-making contexts.

Step 1: Five-Way to Four-Way Reduction This reduction is achieved by setting $\theta_4 = \theta_5$. Since these thresholds become equal, the weak negative and strong negative regions merge, resulting in four-way decisions. The decision rules become:

$$\begin{aligned}
(P''') &: \text{ If } Q(\neg o \mid [S]) \geq \theta_3, \quad S \in POS^{+}(o);\\
(\partial P''') &: \text{ If } Q(\neg o \mid [S]) \geq \theta_3, \quad S \in POS^{-}(o);\\
(\partial N''') &: \text{ If } Q(\neg o \mid [S]) \leq \theta_3, \quad S \in NEG^{-}(o);\\
(N''') &: \text{ If } Q(\neg o \mid [S]) \leq \theta_3, \quad S \in NEG^{+}(o).
\end{aligned}$$

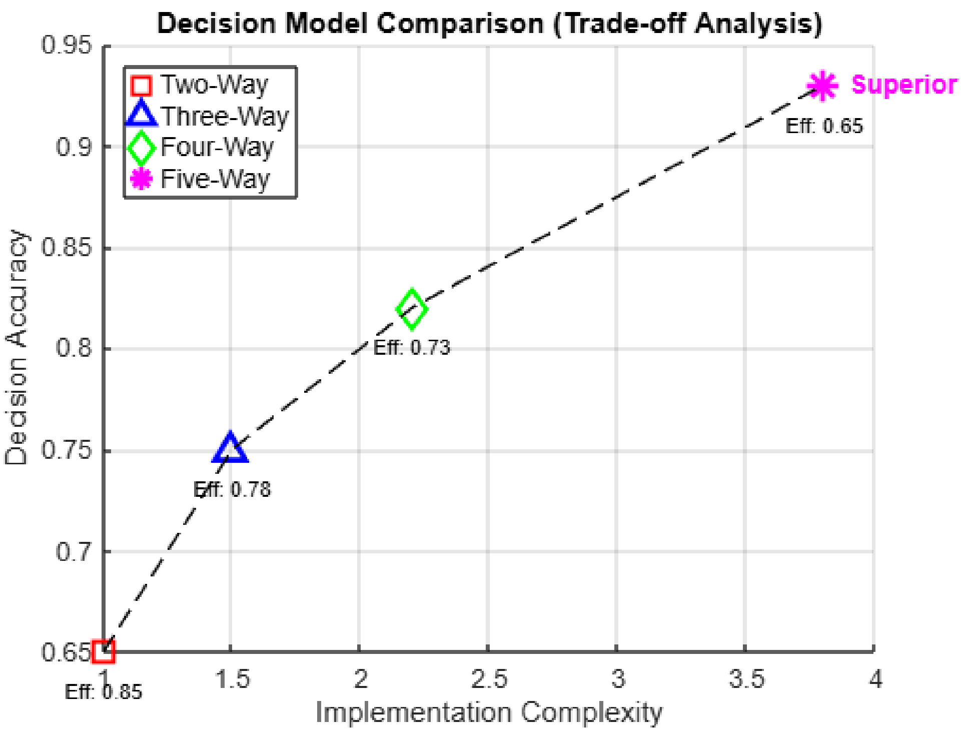

Fig. 2. Comparison of Decision Models

Step 2: Four-Way to Three-Way Reduction This reduction is achieved by setting $\theta_2 = \theta_3$. This causes the weak positive and boundary regions to merge, leading to three-way decisions:

$$\begin{aligned} (P^*) &: \text{ If } Q(\neg o \mid [S]) \geq \theta_2, \quad S \in POS(o); \\ (B^*) &: \text{ If } \theta_1 \leq Q(\neg o \mid [S]) < \theta_2, \quad S \in BND(o); \\ (N^*) &: \text{ If } Q(\neg o \mid [S]) < \theta_1, \quad S \in NEG(o). \end{aligned}$$

Step 3: Three-Way to Two-Way Reduction This reduction is achieved by setting $\theta_1 = \theta_2$. This eliminates the boundary region entirely, resulting in classic two-way decisions:

$$\begin{aligned} (P^\dagger) &: \text{ If } Q(\neg o \mid [S]) \geq \theta_1, \quad S \in POS(o); \\ (N^\dagger) &: \text{ If } Q(\neg o \mid [S]) < \theta_1, \quad S \in NEG(o). \end{aligned}$$

This reduction process simplifies decision-making: 5WD $\rightarrow$ 4WD by merging negative regions, 4WD $\rightarrow$ 3WD by combining weak positive and boundary regions, and 3WD $\rightarrow$ 2WD by eliminating the boundary, leaving only positive and negative regions. This streamlined approach is ideal for binary classification problems requiring a simple accept-or-reject model. Figure 3 represents the reduction of the Five-Way Decision (5WD) model to the Four-Way Decision (4WD) model by simplifying the decision regions and reducing the classification structure. Furthermore, Figure 4 illustrates the transformation of the Four-Way Decision (4WD) model into the Three-Way Decision (3WD) model, highlighting the further simplification of decision boundaries and decision-making regions. Figure 5 presents the reduction of 3WD to 2WD.

6. Formulated Framework

This study introduces a new decision-making model combining MPFNs and a 5-WD Model with a Modified Hamacher Aggregation Operator. The model is designed to offer a strong and adapt-

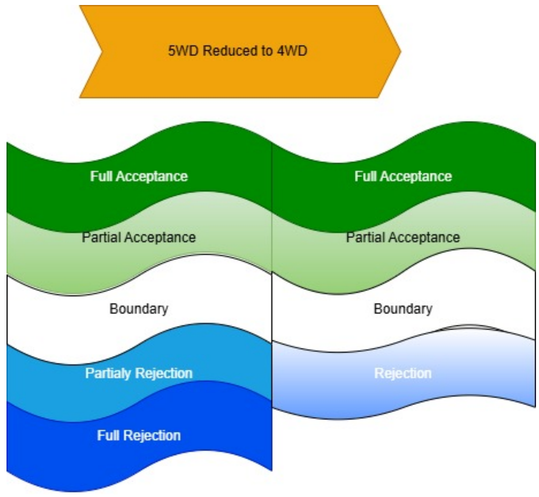

Fig. 3. Reduction of 5WD to 4WD

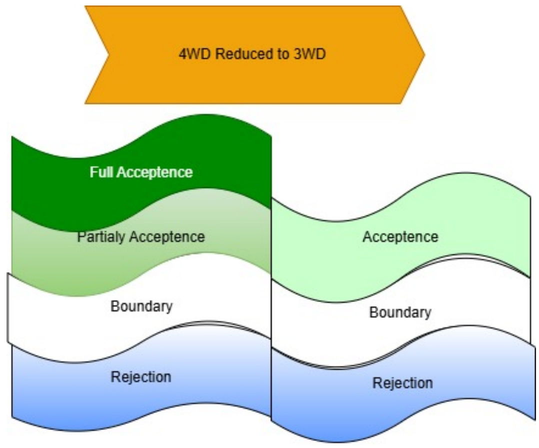

Fig. 4. Reduction of 4WD to 3WD

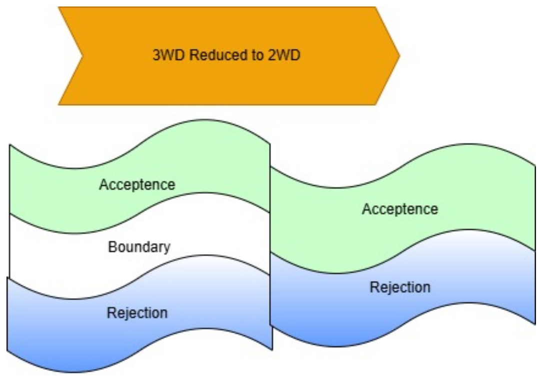

Fig. 5. Reduction of 3WD to 2WD

able methodology for managing complex decision-making situations that are uncertain, as well as in the area of climate change assessment. Let $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m\}$ be a set of alternatives, and $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$ be a set of attributes (criteria). A Modified Hamacher Aggregation Operator is applied to aggregate the MPFNs for each alternative across the attributes, as defined in Eq. (1). The proposed model is structured into the following key steps:
Step 1: Construction of the M-Polar Fuzzy Decision Matrix using expert evaluations for each alterna-

tive under each attribute.
Step 2: Application of the Modified Hamacher Aggregation Operator to combine the polar components of the decision matrix.
Step 3: Computation of Score Functions for each aggregated alternative using a suitable M-Polar Fuzzy Score Function.
Step 4: Implementation of the Five-Way Decision-Making Model, which classifies each alternative into one of the five regions: POS^+, POS^-, BND, NEG^-, NEG^+.
Step 5: Final Ranking and Selection of alternatives based on their aggregated scores and decision regions.
The developed framework allows decision-makers to model uncertainty more effectively, while the incorporation of multiple polarities and the Modified Hamacher Operator ensures flexible aggregation to suit different risk preferences and real-world complexities. Algorithm 7 presents the proposed M-Polar Fuzzy Five-Way Decision-Making procedure using the Modified Hamacher Aggregation operator.

7. A Five-Way Decision-Making Model for Evaluating Carbon Emission Mitigation Strategies Using MPFNs

The 5-WD Model combined with MPFNs provides a powerful and dynamic method of assessing carbon emission mitigation strategies, which is more efficient than the traditional decision-making methods to deal with uncertainty and vagueness. The 5-WD model is more nuanced because, unlike the traditional methods, which usually presuppose binary or three-way choices, it divides the alternatives into five levels: strong acceptance, weak acceptance, boundary (indecision), weak rejection, and strong rejection. Such granularity enables more accurate and realistic evaluation of strategies, particularly in complicated situations such as the mitigation of climate change, where information is frequently inaccurate and evolving.

MPFNs also lead to the further expansion of the model capacity to incorporate numerous dimensions of uncertainty and professional opinions and provide a thorough analysis. MPFNs have a more detailed description of the data, which is compared to other fuzzy-based approaches, e.g., intuitionistic or Pythagorean fuzzy sets, so the model is more flexible in real-world applications. The effectiveness of this approach in the resolution of complex decision-making problems is justified by theoretical groundwork and practical research. In this way, 5-WD not only enhances the accuracy of the decision but also complies with the concepts of sustainability and adaptability and, therefore, is a better alternative to assess the carbon mitigation strategy. [H] [1] Alternatives $\mathcal{A} = \{A_1, \ldots, A_5\}$, weights θ, decision matrix $\mathbf{D}$, parameter $\lambda = 1$ Ranked alternatives with classification

Phase 1: Data Preparation $i = 1$ to 5 $j = 1$ to 5 Verify $\sum_{k=1}^{M}(u_{ij}^k)^3 \leq 1$ $\psi_{ij} \leftarrow \frac{1}{M}\sum_{k=1}^{M} u_{ij}^k$

Phase 2: Hamacher Aggregation $i = 1$ to 5 $\mathcal{N} \leftarrow \prod_{j=1}^{5} \psi_{ij}^{\theta_j}$ $S^{Ham}(A_i) \leftarrow \mathcal{N}$

Phase 3: Score Calculation $i = 1$ to 5 $S(A_i) \leftarrow \sum_{j=1}^{5} \theta_j \psi_{ij}$

Example for A_1**:** $S(A_1) = 0.1(0.8662) + 0.15(0.7261) + 0.2(0.8705) + 0.25(0.6173) + 0.3(0.7817)$ $S(A_1) = 0.7585$

Phase 4: Classification $\tau_1 = 0.8,\ \tau_2 = 0.7,\ \tau_3 = 0.6,\ \tau_4 = 0.5$

$i = 1$ to 5 $S(A_i) \geq \tau_1$ Class $\leftarrow POS^+$ $S(A_i) \geq \tau_2$ Class $\leftarrow POS^-$ $S(A_i) \geq \tau_3$ Class $\leftarrow BND$ $S(A_i) \geq \tau_4$ Class $\leftarrow NEG^-$ Class $\leftarrow NEG^+$

Phase 5: Ranking Sort $\mathcal{A}$ by $S(A_i)$ descending $A^* \leftarrow \arg\max_{1 \leq i \leq 5} S(A_i)$

$\langle A^*$, Class, Ranking$\rangle$

7.1 Five-Way Decision Model Procedure

Suppose we have a set of m alternatives, $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m\}$, and n attributes, $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$. The weight vector for these attributes is $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, where each $\mu_j \in [0, 1]$ and $\sum_{j=1}^{n} \mu_j = 1$.

Each evaluation $\mathcal{S}_{ij} = (u_{\mathcal{S}_{ij}}, v_{\mathcal{S}_{ij}})$ represents the assessment of alternative $\mathcal{S}_i$ under attribute $\mathcal{G}_j$ and is expressed as an M-Polar Fuzzy Number, satisfying $u_{\mathcal{S}_{ij}}, v_{\mathcal{S}_{ij}} \in [0, 1]$ with $u^2_{\mathcal{S}_{ij}} + v^2_{\mathcal{S}_{ij}} \leq 1$.

Given the attribute characteristics, decision-makers determine a risk avoidance coefficient $\Re_j$ for each attribute $\mathcal{G}_j$. This information forms the decision matrix $[\mathcal{S}_{ij}]_{m \times n}$, from which we make decisions for each alternative. The main steps in this Five-Way Decision Model are:
Step 1: Construct a comparative decision matrix to evaluate each alternative against the five attributes.
Step 2: Use MPFNs to model uncertainty and vagueness in the data.
Step 3: Apply the Modified Hamacher Aggregation Operator to aggregate scores and derive a comprehensive evaluation.
Step 4: Rank the alternatives using a utility function to identify the most effective and feasible strategies.

8. Application: Evaluating Carbon Emission Mitigation Strategies

The issue of climate change is an international crisis, and carbon emissions are a significant factor in the intensification of global climate change. The main sources of these emissions are industrial processes, the use of fossil fuels, deforestation, and other human activities, which cause increasing global temperatures, severe weather conditions, and ecological disturbances. In order to reduce these negative effects, policymakers and environmental scientists should consider various options and choose the most efficient. Nevertheless, in this case, the process of decision-making is complicated by uncertainties, inaccurate information, and the dynamism of climate-related variables. To solve these problems, MPFNs provide a mathematical system that is structured and able to deal with vagueness and uncertainty in decision making.. By using this method, a more accurate assessment of strategies can be made by considering numerous views and uncertainty levels, as shown in Figure 6. To evaluate

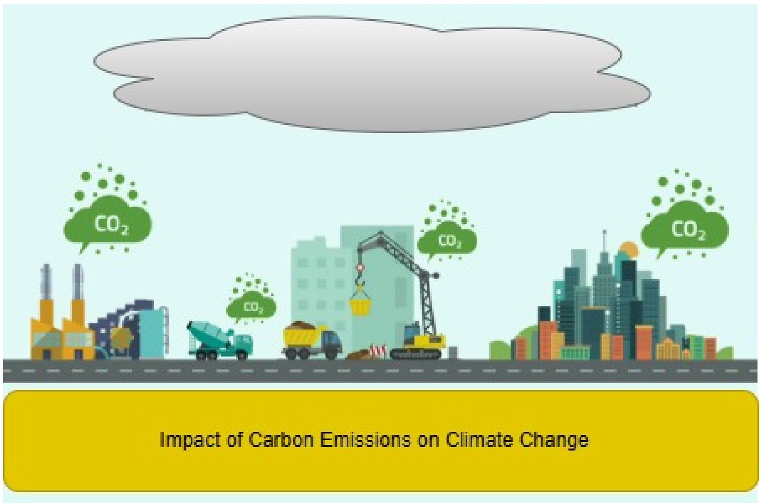

Fig. 6. Carbon Emission Sources

in a systematic way different methods of carbon emission reduction, we specify important aspects of alternatives (mitigation strategies) and attributes (evaluation criteria), as presented in Table 5. A total of five major alternatives are used to evaluate the carbon emission mitigation strategies, where each alternative represents a different method of mitigating carbon emissions.

Table 5
Alternatives and Evaluation Criteria for Carbon Mitigation Strategies

Alternatives (Strategies)	Description
A_1	Renewable Energy Adoption
A_2	Carbon Capture Technologies
A_3	Regulatory Policies
A_4	Afforestation
A_5	Sustainable Transportation
Attributes (Criteria)	**Description**
C_1	Environmental Impact
C_2	Technological Feasibility
C_3	Cost-Effectiveness
C_4	Scalability
C_5	Implementation Time

The evaluation model would be developed around five major properties that holistically reflect the performance of both alternatives. Environmental Impact (C_1) is an indicator of how well a strategy can be used to decrease greenhouse gas emissions and also the overall benefit of the strategy to the environment. Technological Feasibility (C_2) is an evaluation of the technical maturity, reliability, and flexibility of the proposed solution. Cost-Effectiveness (C_3) is a measure of the economic feasibility based on the initial investment and long-term efficiency of the operation. Scalability (C_4)investigates the degree to which a strategy can be extended and applied in the various regions and sectors. Lastly, Implementation Time (C_5) is the time that is taken to deploy and actualize anticipated benefits, which is essential to make timely decisions in dynamic environmental situations.

Assigning Weights to Criteria

We assign weights to each criterion to reflect its relative importance. The weights are normalized such that they sum to 1. For this study, the weight vector is defined as:

$$\mathbf{W} = (w_1, w_2, w_3, w_4, w_5),$$

where the values are presented in Table 6.

Table 6
Weights Assigned to Evaluation Criteria

Weight	Criteria	Value
w_1	Environmental Impact	0.30
w_2	Technological Feasibility	0.25
w_3	Cost-Effectiveness	0.20
w_4	Scalability	0.15
w_5	Implementation Time	0.10

The weights satisfy $\sum_{i=1}^{5} w_i = 1$. Environmental impact is assigned the highest weight (0.30), reflecting its critical importance in carbon mitigation decisions, while implementation time receives the lowest weight (0.10).

Decision Matrix and Aggregation

Each alternative is evaluated using MPFNs across the five criteria. Table 7 presents the initial decision matrix.

Table 7
Decision Matrix Using MPFNs for Carbon Mitigation Strategies

Alternatives	C_1	C_2	C_3	C_4	C_5
A_1	(0.85,0.75,0.90,0.70,0.80)	(0.80,0.70,0.85,0.65,0.75)	(0.78,0.68,0.80,0.60,0.72)	(0.88,0.78,0.92,0.72,0.82)	(0.82,0.72,0.88,0.68,0.78)
A_2	(0.78,0.68,0.82,0.62,0.72)	(0.75,0.65,0.80,0.60,0.70)	(0.72,0.62,0.78,0.58,0.68)	(0.80,0.70,0.85,0.65,0.75)	(0.75,0.65,0.80,0.60,0.70)
A_3	(0.90,0.80,0.95,0.75,0.85)	(0.88,0.78,0.92,0.72,0.82)	(0.85,0.75,0.90,0.70,0.80)	(0.92,0.82,0.96,0.76,0.86)	(0.88,0.78,0.92,0.72,0.82)
A_4	(0.82,0.72,0.88,0.68,0.78)	(0.80,0.70,0.85,0.65,0.75)	(0.78,0.68,0.80,0.60,0.72)	(0.85,0.75,0.90,0.70,0.80)	(0.80,0.70,0.85,0.65,0.75)
A_5	(0.70,0.60,0.75,0.55,0.65)	(0.68,0.58,0.72,0.52,0.62)	(0.65,0.55,0.70,0.50,0.60)	(0.72,0.62,0.78,0.58,0.68)	(0.68,0.58,0.72,0.52,0.62)

The aggregated score $S(A)$ for each alternative is calculated using the Modified Hamacher Aggregation Operator:

$$S(A) = \frac{\lambda \prod_{i=1}^{m} \psi_i^{\theta_i}}{\lambda + (1-\lambda)\left(\sum_{i=1}^{m} \theta_i \cdot \psi_i - \prod_{i=1}^{m} \psi_i\right)}, \tag{16}$$

where λ is the parameter controlling the aggregation behavior, θ_i represents the weights, and ψ_i denotes the membership values. For simplicity, we set $\lambda = 1$, which reduces the operator to the weighted geometric mean:

$$S(A) = \prod_{i=1}^{m} \psi_i^{\theta_i}.$$

Applying this operator yields the aggregated weights presented in Table 8.

Table 8
Aggregated Matrix Using MPFNs

Alternatives	**Aggregated Weight Values**
A_1	(0.8662, 0.7261, 0.8705, 0.6173, 0.7817)
A_2	(0.7589, 0.6590, 0.8100, 0.6099, 0.7100)
A_3	(0.8881, 0.7856, 0.9288, 0.7287, 0.8288)
A_4	(0.8101, 0.7100, 0.8548, 0.6546, 0.7589)
A_5	(0.6856, 0.5856, 0.7424, 0.5332, 0.6333)

Using the score function defined in Eq. (5), we calculate the overall score for each alternative:

$$S_i(A_i) = \frac{1}{i} \sum_{k=1}^{m} (\theta_k \circ \psi_A). \tag{17}$$

For A_1

$$S_1(A_1) = 0.1 \times 0.8662 + 0.15 \times 0.7261 + 0.2 \times 0.8705 + 0.25 \times 0.6173 + 0.3 \times 0.7817 = 0.7585.$$

For A_2

$$S_2(A_2) = \frac{1}{2}(0.1 \times 0.7589 + 0.15 \times 0.6590 + 0.2 \times 0.8100 + 0.25 \times 0.6099 + 0.3 \times 0.7100) = 0.3511.$$

The complete score matrix is presented in Table 9.

Table 9
Score Matrix of Alternatives

Alternatives	Score Value
A_1	0.7585
A_2	0.3511
A_3	0.2744
A_4	0.1946
A_5	0.1418

Using the threshold values defined in Table 10, we classify each alternative into one of the five decision regions.

Table 10
Threshold Values for Decision Regions

Threshold	Value
θ_5 (Strong Acceptance)	0.30
θ_4 (Weak Acceptance)	0.25
θ_3 (Boundary)	0.20
θ_2 (Weak Rejection)	0.15
θ_1 (Strong Rejection)	0.10

The classification results are obtained using the defined threshold-based decision framework, leading to a structured allocation of alternatives across different decision regions. The results indicate that alternatives A_1 and A_2 fall within the strong acceptance (POS^+) region, as their values (0.7585 and 0.3511) both exceed the threshold of 0.30, demonstrating their high suitability and priority among the available options.

Furthermore, alternative A_3 is categorized under weak acceptance (POS^-), with its value (0.2744) lying between 0.25 and 0.30, indicating a moderate level of acceptance. Alternative A_4 is placed in the boundary (BND) region, as its score (0.1946) falls between 0.15 and 0.20, reflecting uncertainty in decision-making. In addition, alternative A_5 is classified under weak rejection (NEG^-), since its value (0.1418) lies between 0.10 and 0.15, suggesting relatively low preference. Notably, no alternative falls within the strong rejection (NEG^+) region, indicating that none of the options are entirely unsuitable under the given evaluation criteria. Figure 7 represents the ranking of alternatives obtained from the proposed decision-making approach.

8.1 *Summary of Sub-Criteria Evaluations*

Table 11 presents the comparison of sub-criteria for Criteria C_1 (Temperature Rise).

Table 11
Comparison of Sub-Criteria for Temperature Rise

Sub-Criterion	Description	Example (A_1)	Example (A_3)
c_1	Likelihood of temperature rise	High (0.8)	Low (0.3)
c_2	Magnitude of temperature rise	Large (0.7)	Small (0.4)
c_3	Uncertainty in temperature projections	High (0.6)	Low (0.2)
c_4	Impact on regional temperature variations	Significant (0.9)	Minimal (0.5)
c_5	Long-term temperature stabilization potential	Low (0.3)	High (0.7)

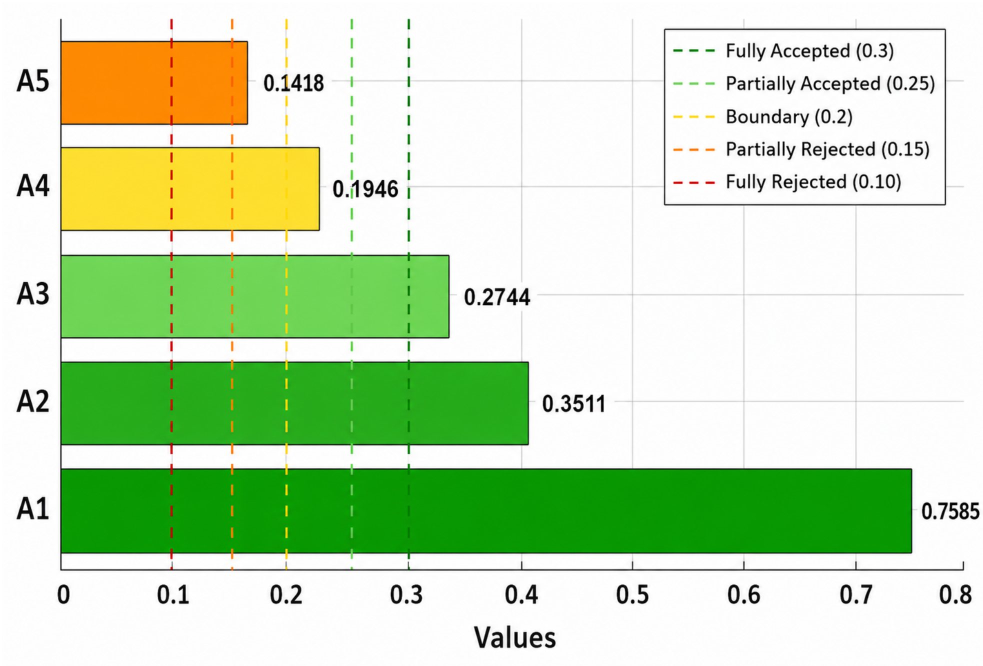

Fig. 7. Ranking of Alternatives

Table 12 presents the decision matrix for Criteria C_2 (Sea-Level Rise).

Table 12
Decision Matrix for Sea-Level Rise

	C_1	C_2	C_3	C_4	C_5
A_1	$(0.7, 0.6, 0.8, 0.5, 0.3)$	$(0.8, 0.5, 0.6, 0.7, 0.2)$	$(0.6, 0.8, 0.4, 0.5, 0.6)$	$(0.5, 0.7, 0.6, 0.8, 0.3)$	$(0.7, 0.6, 0.8, 0.5, 0.3)$
A_2	$(0.6, 0.7, 0.5, 0.8, 0.4)$	$(0.7, 0.6, 0.8, 0.5, 0.3)$	$(0.8, 0.5, 0.6, 0.7, 0.2)$	$(0.6, 0.8, 0.4, 0.5, 0.6)$	$(0.5, 0.7, 0.6, 0.8, 0.3)$
A_3	$(0.8, 0.5, 0.6, 0.7, 0.2)$	$(0.6, 0.8, 0.4, 0.5, 0.6)$	$(0.5, 0.7, 0.6, 0.8, 0.3)$	$(0.7, 0.6, 0.8, 0.5, 0.3)$	$(0.8, 0.5, 0.6, 0.7, 0.2)$
A_4	$(0.5, 0.7, 0.6, 0.8, 0.3)$	$(0.7, 0.6, 0.8, 0.5, 0.3)$	$(0.8, 0.5, 0.6, 0.7, 0.2)$	$(0.6, 0.8, 0.4, 0.5, 0.6)$	$(0.5, 0.7, 0.6, 0.8, 0.3)$
A_5	$(0.7, 0.6, 0.8, 0.5, 0.3)$	$(0.8, 0.5, 0.6, 0.7, 0.2)$	$(0.6, 0.8, 0.4, 0.5, 0.6)$	$(0.5, 0.7, 0.6, 0.8, 0.3)$	$(0.7, 0.6, 0.8, 0.5, 0.3)$

Applying the Modified Hamacher Aggregation Operator yields the aggregated matrix in Table 13.

Table 13
Aggregated Matrix for Sea-Level Rise

Alternatives	**Aggregated Weight Values**
A_1	(0.6366, 0.6427, 0.6274, 0.5915, 0.3243)
A_2	(0.6158, 0.6612, 0.5558, 0.6454, 0.3385)
A_3	(0.6746, 0.6427, 0.6006, 0.6285, 0.2830)
A_4	(0.6046, 0.6611, 0.5661, 0.5054, 0.3289)
A_5	(0.6366, 0.6426, 0.6207, 0.5914, 0.3242)

The score values for sea-level rise criteria are

$$S(A_1) = 0.53071, \quad S(A_2) = 0.2674, \quad S(A_3) = 0.1753, \quad S(A_4) = 0.1242, \quad S(A_5) = 0.1058.$$

The classification for sea-level rise is derived using the same evaluation framework, resulting in a clear categorization of alternatives based on their respective scores and threshold limits. The results show that alternative A_1 lies in the strong acceptance region, as its value (0.53071) exceeds the threshold of 0.30, indicating the highest level of suitability among all alternatives.

In addition, alternative A_2 is classified under weak acceptance, with its value (0.2674) falling between 0.25 and 0.30, reflecting moderate performance. Alternative A_3 is positioned in the boundary region, as its score (0.1753) lies between 0.15 and 0.20, indicating a level of uncertainty in decision-making. Finally, alternatives A_4 and A_5 fall into the weak rejection category, since their values (0.1242 and 0.1058) are within the range of 0.10 to 0.15, suggesting comparatively lower preference. Table 14 illustrates the evaluation of sea-level rise sub-criteria under Criterion C_2.

Table 14
Summary of Criteria for Sea-Level Rise

Sub-Criterion	Description	Example (A_1)	Example (A_3)
c_1	Likelihood of sea-level rise	High (0.9)	Low (0.4)
c_2	Magnitude of sea-level rise	Large (0.8)	Small (0.3)
c_3	Impact on coastal regions	Significant (0.7)	Minimal (0.5)
c_4	Uncertainty in sea-level projections	High (0.6)	Low (0.2)
c_5	Long-term sea-level stabilization potential	Low (0.5)	High (0.7)

Criteria C_3 (Extreme Weather Events) focuses on the analysis of extreme climatic variations such as floods, storms, heatwaves, and droughts affecting the system. The complete evaluation for extreme weather events follows the same methodological procedure, yielding a structured classification of alternatives based on their computed scores and predefined threshold values. The results indicate that alternative A_1 falls within the strong acceptance region, as its value (0.52087) exceeds the upper threshold of 0.30. This reflects a highly favorable assessment and suggests that A_1 is the most suitable option under the given decision framework.

Furthermore, alternative A_2 is categorized under weak acceptance, with its value (0.2653) lying between 0.25 and 0.30, indicating moderate suitability. Alternative A_3 is placed in the boundary region, as its score (0.1844) falls between 0.15 and 0.20, representing uncertainty in decision-making. Finally, alternatives A_4 and A_5 are classified under weak rejection, since their values (0.1302 and 0.1060) lie between 0.10 and 0.15, suggesting relatively lower preference compared to other alternatives. Criteria C_4 (Economic Costs of Mitigation) describes the analysis of economic feasibility and cost-effectiveness of mitigation actions. The classification results for economic costs are:

a. **Strong Acceptance**: A_1 (0.6523 ¿ 0.30), A_2 (0.3303 ¿ 0.30)

b. **Boundary**: A_3 ($0.20 > 0.1742 > 0.15$)

c. **Weak Rejection**: A_4 ($0.15 > 0.1256 > 0.10$), A_5 ($0.15 > 0.1084 > 0.10$)

8.2 *Comparison Between Three-Way and Five-Way Decision Models*

Table 15 presents a comparative analysis between the Three-Way Decision model and the Five-Way Decision model in terms of regions, thresholds, complexity, and application scenarios.

Table 15
Comparison of Three-Way Decision and Five-Way Decision Models

Feature	3-WD Model	5-WD Model
Regions	3 (Accept/Reject/Boundary)	5 (Strong Accept/Weak Accept/Boundary/Weak Reject/Strong Reject)
Thresholds	2 (θ_1, θ_2)	4 ($\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$)
Complexity	Low (simpler rules)	High (nuanced overlapping regions)
Use Case	Scenarios needing rapid decisions	Highly uncertain, multi-stage decisions

Figure 8 illustrates the validity comparison between the Three-Way Decision and Five-Way Decision models.

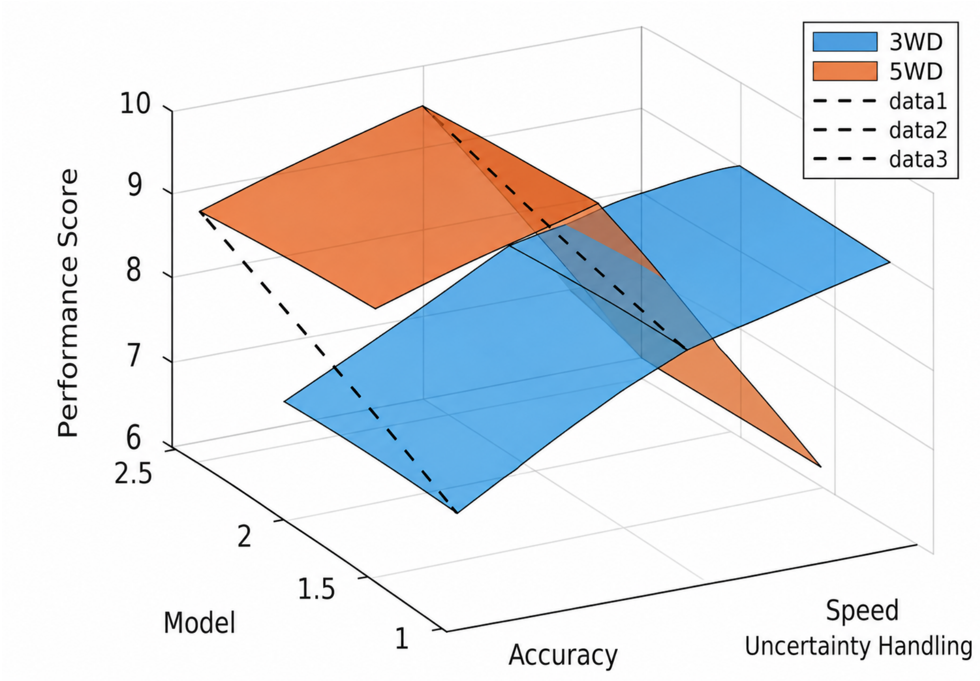

Fig. 8. Validity Comparison Between 3WD and 5WD

9. Conclusion

The study has come up with a novel framework of decision-making that is able to blend MPFNs and a 5-WD model with the benefit of a Modified Hamacher Aggregation Operator to deal with the multidimensional problem of mitigating climate change. The proposed model is more suitable for multi-dimensional uncertain situations and an organised method of assessing the carbon reduction strategies. The framework provides policymakers with a highly effective means of determining the best mitigation strategies by considering the complexities of the real world and the unavailability of information through its strong classification process and the broad-based ranking system.

The research contributes a lot to the theoretical and applied fields of environmental decision-making. Besides advancing the fuzzy set theory by introducing new methods of aggregation, the developed methodology also provides a viable solution to climate policy-making. The analysis of carbon mitigation programs in various criteria, such as temperature rise, sea-level rise, extreme weather events, and economic costs, proves the flexibility and efficiency of the framework. Further study ought to include the application of machine learning methods and the implementation of the framework into other areas of sustainability, including air pollution control, water resource management, and environmental sustainability. The work provides a solid base of data-driven environmental governance and new possibilities of creating intelligent decision-support systems in sustainability science.

Future Work

The study can be expanded further through future research, which should create a powerful MPFN-based decision-making model to be combined with the Modified Hamacher Aggregation Operator to examine the effects of fossil fuel emissions on climate change in more detail. Moreover, a thorough ranking system within the framework of 5-WD can be optimized to categorize alternatives into several acceptance and rejection areas of a better quality and clarity. In addition, further research can be done on how to enrich decision support systems for policymakers with multi-criteria and multi-dimensional frameworks of fuzzy logic, which can deal with high rates of uncertainty in climate-related data. A comparative analysis of the different emission reduction strategies can also be done in detail to determine the most effective and sustainable ways of reducing the effects of climate change.

The interdependencies of climate variables and fossil fuel emissions can be further investigated by applying the sophisticated M- Polar fuzzy aggregation methods, which will result in more analytical information. Scientists can also come up with systematic approaches to assessing renewable energy uptake and the reduction of carbon policy in uncertain conditions. Furthermore, the proposed model may be used on real-world data to prove the effectiveness and the possibility of practical application in resolving climate-related issues. Aggregation operators can be extended to address more complex and dynamic decision-making problems, which can be considered as theoretical developments in MPFNs.

Lastly, the work of the future can be oriented on creating a flexible and generalized framework that can be deployed to other sustainability areas, including air pollution control, water resource management, and environmental protection. Another opportunity to make M-Polar fuzzy decision-making models more robust and efficient is the integration of hybrid fuzzy methods with artificial intelligence and machine learning techniques.

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Conflicts of Interest
The authors declare no conflicts of interest.

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