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Efficient Solution Strategy for Transportation Linear Fractional Programming under Fuzzy Conditions

Sultan S. Alodhaibi[1], Moodi Abdulrahman Abdullah Al- Rajeh[1], Florentin Smarandache[2], Hamiden Abd El- Wahed Khalifa[1,3,*]

[1] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia
[2] Mathematics, Physics and Natural Science Division, The University of New Mexico, Gallup, MN, USA
[3] Department of Operations and Management Research, Faculty of Graduate Studies and Statistical Research, Cairo University, Giza 12613, Egypt

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ABSTRACT

Linear fractional programming (LFP) serves as an effective optimization framework in which the objective function is formulated as the ratio of two linear expressions. In numerous daily-life situations, the parameters involved in the objective function may not be defined precisely, leading to the incorporation of fuzzy representations. Transportation problems with fractional objectives are commonly described through system representations involving a limited number of lumps interconnected by curves. This study investigates a transportation model with a linear fractional objective where the parameters are characterized using fully piecewise quadratic fuzzy (PQF) numbers (PQFNs). To address the inherent uncertainty, a closeness-based interval approximation of these fuzzy numbers is introduced, allowing the original problem to be converted into an interval-valued LFP model. By applying the Charnes–Cooper transformation, this formulation is further reduced to an interval-valued linear transportation problem. This is then converted into a multi-objective optimization problem. Order relations on interval-valued outcomes are used to represent preferences of decision makers, considering lower and upper bounds, midpoints, and interval widths. An appropriate weighting scheme is then used to arrive at an ideal cooperation outcome. An example of the applicability and computational efficiency of the proposed approach is illustrated mathematically. Concluding the paper, the main findings and possible areas for further research are given.

1. Introduction

LFP is now a popular optimization methodology to model and solve complex problems in areas like production planning, finance, healthcare and engineering. The objective function, in this context, is formulated as a ratio of two linear functions which permits a more realistic representation of efficiency-based performance measures. FTP is another important application of LFP with a major role in the management of logistics and supply chain through the facilitation of cost reduction and

* *Corresponding author.*
E-mail address: author.mail@gmail.com

facility enhancement. The classical transportation problem can be commonly modeled as a network based on a finite set of destinations and sources, which are interconnected by arcs. The objective function in FTP is in a fractional form, thus the model is more appropriate to determine system performance in realistic situations. The preliminary version of FTP was proposed by Swarup [1]. This notion was later developed by Joshi and Gupta [2] who took into account the variability in supply and demand and suggested a methodology that gives the range of solutions, thus being more useful in the decision making environment. To tackle transportation issues in deterministic environments, all parameters are assumed to be known, and over the years, a number of solution methods have been suggested. But in reality, other parameters like transportation costs, supply capacities, and the level of demand are unpredictable or inaccurate. These uncertainties can be caused by the changing market conditions, changing relationships in supply chain or lack of complete information. As a result, there has been an interest in modeling uncertainty in transportation. A number of researchers have studied linear FTPs in the uncertain setting. As an example, Safi and Ghasemi [3] investigated LFTPP under uncertain parameters in terms of the expected value of the objective function and the uncertainty constraints. On the same note, Mahmoodirad *et al.,* [4] used chance-constrained programming to deal with uncertainty in LFTPP. Sheikhi and Ebadi [5] analyzed fuzzy LFTPP through the conversion of fuzzy parameters to crisp ones based on ranking functions and employed strong ranking methods to find the best solutions. In another work, Sheikhi and Ebadi [6] concentrated on interval-valued FTPs and suggested an algorithm which would combine interval analysis and optimization techniques to obtain credible solutions.

More developments can be seen with the Mehar method suggested by Bhatia *et al.,* [7] to solve LFTPP. Akram *et al.,* [8] proposed an interval-valued Fermatean fuzzy fractional transportation model and proposed a solution methodology without transforming the model into a crisp counterpart. Khalifa *et al.,* [9] studied FTPs in a neutrosophic setting, with uncertainty in terms of supply, demand, and transportation costs. Anukokifa *et al.,* [10] tackled the multi-objective fractional stochastic solid transportation issues based on the Weibull distribution and reformulated them as goal programming models under probabilistic constraints. Moreover, Joshi *et al.,* [11] examined multi-choice solid fractional multi-objective transportation problems with the uncertainty of the parameters addressed by interpolation and ranking procedures. Mohanaselvi and Ganesan [12] suggested solution procedures of fuzzy FTPs by generalizing classical procedures like the approximation of Vogel and the MODI method to fuzzy environment. To improve the decision-making within logistics systems, Sheikhi *et al.,* [13] designed a methodology that could be used to get effective solutions in bi-objective fuzzy FTPs based on ranking functions. In addition, Joshi *et al.,* [14] developed a parametrized generalized method of solving fuzzy multi-objective transportation problems and proved to be efficient by numerical case studies and comparisons with other methods, like fuzzy DEA and grey relational analysis. Agrawal and Ganesh [15] studied random multi-choice transportation problems and suggested the techniques of resolving optimal solutions in case of uncertainty. The multi-objective linear FTP (LFTP) developed by Saini *et al.,* [16] used different methods, such as weighted sum and fuzzy programming to give compromise solutions. Revathi and Mohanaselvi [16] have done an extension of this by taking into account a four-dimensional multi-item multi objective multi-dimensional transportation model with uncertain parameters. Garg *et al.,* [17] worked on fuzzy fractional two-stage transshipment problems and used the α-level decomposition and Charnes-Cooper transformation to deal with the nonlinearity in the objective function.

Although these have been significant contributions, a number of gaps in research still exist. The fuzzy representations used in most of the existing studies are simple ones e.g. triangular or trapezoidal fuzzy numbers, which are not necessarily sufficient to represent the intricate patterns of uncertainty. PQFNs use in the linear FTPs has not been well exploited. Moreover, effective methodologies of integrating PQFNs directly into optimization models are lacking since most of the current methods reduce the problem to simple fuzzy structures. The other constraint is the transformation of such models. Even though interval-valued methods are widespread, there has been little research on how to develop strong methods to transform PQFNs into interval representations. Specifically, interval approximation methods that are based on proximity have not been thoroughly explored. Also, although the Charnes& Cooper transformation [18] is a long-established method to transform fractional programming (FP) problems (FPPs) to linear forms, the extension of the method to interval-valued models based on advanced fuzzy representations is not well developed. Furthermore, most interval-based optimization techniques lack an adequate consideration of preferences of decision-makers. In practice, decisions are made based on how interval values are interpreted, their lower and upper bounds, midpoints and widths. The lack of extensive order relations based on these features reduces the relevance of current models. Multi-objective frameworks based on interval-valued transportation problems based on fuzzy fractional formulations are also lacking. Equally, the use of weighting methods in producing efficient (Pareto-optimal) solutions in these situations is a relatively unexplored area. Lastly, the number of numerical validation of the existing works is limited, which diminishes the possibility of evaluating the efficiency and computational efficiency of suggested approaches.

In order to circumvent these restrictions, the current research suggests a new model to solve linear FTPs in uncertainty. The proposed method uses PQFNs in the objective function to capture complex uncertainty. The interval-based approximation of fuzzy parameters is presented where a proximity-based approximation is used to convert the parameters into interval forms that are solvable. The resulting model is then Charnes & Cooper transformed to make the resulting model equivalent to a linear transportation problem. Moreover, the reformulated model is presented in the form of a multi-objective optimization problem, with the preferences of the decision-maker integrated in the form of order relations, using interval properties like lower bounds, upper bounds, midpoints, and widths. A weighting approach is used to get effective solutions that will balance a variety of criteria. The validity and usefulness of the suggested methodology are illustrated by a numerical example.

The rest of this paper will be structured as below. Section 2 defines and provides arithmetic operations of PQFNs. Section 3 develops the LFTP using PQFNs. Section 4 outlines the solution procedure proposed. A numerical example is given in Section 5. Section 6 explains the strengths and weaknesses of the methodology and Section 7 finally wraps up the study by research directions.

2. Preliminaries

In this section, some definitions related to the PQFNs, interval- valued approximation, and their arithmetic operations are introduced.

Definition 1. [19] Let Hdenote the universe of discourse and let $\zeta \in H$be an arbitrary element. A fuzzy set (FS) $\tilde{A}$defined on $H$is expressed as: $\tilde{A} = \{ x \in H \mid \mu_{\tilde{A}}(x) \}$,where the function $\mu_{\tilde{A}}: H \to [0,1]$signifies the association grade of each element $x \in H$, filling $0 \leq \mu_{\tilde{A}}(x) \leq 1$. Here, $\mu_{\tilde{A}}(x)$ is referred to as the association function of the FS $\tilde{A}$.

Definition 2. [20] Let $\mathbb{R}$ be the set of finite elements and consider a FS $\tilde{A}$ defined by the mapping $\mu_{\tilde{A}}: \mathbb{R} \to [0,1]$. The set $\tilde{A}$ is called a fuzzy number if it contents the subsequent situations:

There exists at least one element $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_0) = 1$.

For any $x, y \in \mathbb{R}$ and $\gamma \in [0,1]$, $\mu_{\tilde{A}}(\gamma x + (1-\gamma) y) \geq \min\big(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\big)$.

The membership function $\mu_{\tilde{A}}(x)$ is piecewise continuous on $\mathbb{R}$.

Definition 3. [21] A PQFN, denoted by $\tilde{A}^{PQFN} = (a_1, a_2, a_3, a_4, a_5)$, is defined on the real line such that$a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$, where all $a_i \in \mathbb{R}$. The membership function $\mu_{\tilde{A}^{PQFN}}(x)$ is described by a piecewise quadratic form over the interval $[a_1, a_5]$. Graphically, the PQFN exhibits a smooth, curved shape characterized by quadratic segments (see Figure 1). The membership value starts from zero at $x = a_1$, increases gradually in a convex quadratic manner up to the left shoulder region near a_2, and continues rising until it reaches its maximum value around the central point a_3. In this central region, the membership function may attain or approach unity, representing the highest degree of belonging. Beyond a_3, the membership degree decreases symmetrically (or near-symmetrically) in a concave quadratic fashion through the right shoulder region near a_4, eventually reaching zero at $x = a_5$. Thus, the function is continuous and piecewise quadratic over its domain, with a single peak and smooth transitions between intervals. The parameters a_1 and a_5 define the support of the fuzzy number, where the membership degree is zero, while a_2, a_3, and $a_4$determine the shape and spread of the curve, including the core and transition regions.

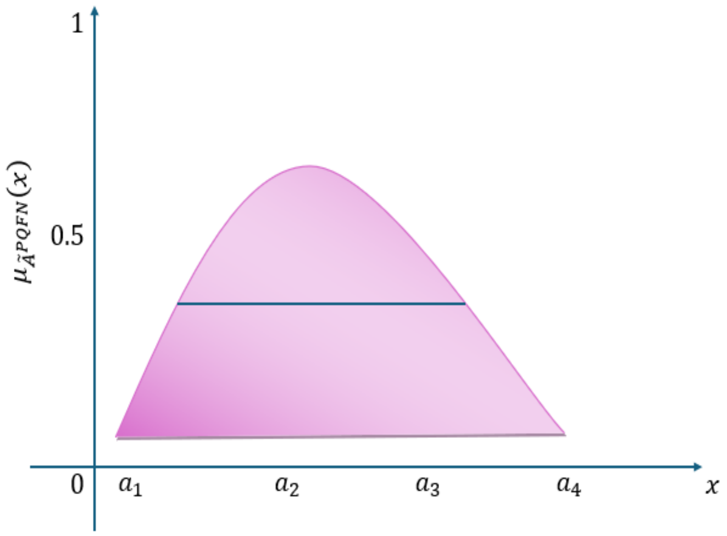

Fig. 1. Visual representation of the membership function of a PQFN.

Definition 4. [21] Let $\tilde{A}^{PQFN} = (a_1, a_2, a_3, a_4, a_5)$and $\tilde{B}^{PQFN} = (b_1, b_2, b_3, b_4, b_5)$be two PQFNs. The basic arithmetic operations between these PQFNs are defined component-wise as follows:

i. Addition: $\tilde{A}^{PQFN} \oplus \tilde{B}^{PQFN} = (a_1 + b_1, \dots, a_5 + b_5)$.
ii. Subtraction: $\tilde{A}^{PQFN} \ominus \tilde{B}^{PQFN} = (a_1 - b_5,\ a_2 - b_4,\ a_3 - b_3,\ a_4 - b_2,\ a_5 - b_1)$.
iii. Scalar Multiplication: For a scalar $k \in \mathbb{R}$, $k\tilde{A}^{PQFN} = \begin{cases} (ka_1,\ ka_2,\ ka_3,\ ka_4,\ ka_5), & \text{if } k > 0, \\ (ka_5,\ ka_4,\ ka_3,\ ka_2,\ ka_1), & \text{if } k < 0. \end{cases}$
iv. Maximum: $\tilde{A}^{PQFN} \vee \tilde{B}^{PQFN} = (\max(a_1, b_1), \dots, \max(a_5, b_5))$.
v. Minimum: $\tilde{A}^{PQFN} \wedge \tilde{B}^{PQFN} = (\min(a_1, b_1), \dots, \min(a_5, b_5))$.

Definition 5. [21] Let $\tilde{A}$ be a PQFN. An interval representation of $\tilde{A}$, denoted by $[A] = [a_\alpha^-, a_\alpha^+]$, is referred to as a closed interval approximation if its bounds are determined based on the $\alpha = 0.5$level set of the membership function. Specifically, $a_\alpha^- = \inf\{ x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) \geq 0.5 \}, a_\alpha^+ = \sup\{ x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) \geq 0.5 \}$. In other words, the interval $[a_\alpha^-, a_\alpha^+]$ represents all real values whose membership degree is at least 0.5, thereby capturing the central region of the fuzzy number.

Definition 6. [21] Let $[A] = [a_\alpha^L, a_\alpha^U]$ denote the closed interval approximation of a PQFN. The corresponding associated crisp value (or representative scalar) of this interval is defined as $\hat{A} = \frac{a_\alpha^L + a_\alpha^U}{2}$. This value represents the midpoint of the interval and provides a single numerical estimate that captures the central tendency of the fuzzy number.

Definition 7. [21] Let $[A] = [a_\alpha^-, a_\alpha^+]$and $[B] = [b_\alpha^-, b_\alpha^+]$be two interval approximations corresponding to PQFNs. The fundamental arithmetic operations on these intervals are defined as follows:

Algebraic Operations

i. Addition: $[A] \oplus [B] = [\, a_\alpha^- + b_\alpha^-,\ a_\alpha^+ + b_\alpha^+ \,]$.
vi. Subtraction:$[A] \ominus [B] = [\, a_\alpha^- - b_\alpha^+,\ a_\alpha^+ - b_\alpha^- \,]$.
vii. Scalar Multiplication: For $k \in \mathbb{R}$, $k[A] = \begin{cases} [\, ka_\alpha^-,\ ka_\alpha^+ \,], & k > 0, \\ [\, ka_\alpha^+,\ ka_\alpha^- \,], & k < 0. \end{cases}$
viii. Multiplication: $[A] \otimes [B] = \left[\frac{a_\alpha^+ b_\alpha^- + a_\alpha^- b_\alpha^+}{2},\ \frac{a_\alpha^- b_\alpha^- + a_\alpha^+ b_\alpha^+}{2}\right]$.
ix. Division: $\frac{[A]}{[B]} = \begin{cases} \left[\frac{2a_\alpha^-}{b_\alpha^- + b_\alpha^+},\ \frac{2a_\alpha^+}{b_\alpha^- + b_\alpha^+}\right], & [B] > 0,\ b_\alpha^- + b_\alpha^+ \neq 0, \\ \left[\frac{2a_\alpha^+}{b_\alpha^- + b_\alpha^+},\ \frac{2a_\alpha^-}{b_\alpha^- + b_\alpha^+}\right], & [B] < 0,\ b_\alpha^- + b_\alpha^+ \neq 0. \end{cases}$

Lattice Operations

i. Minimum: $[A] \wedge [B] = [\ \min(a_\alpha^-, b_\alpha^-)\ ,\ \min(a_\alpha^+, b_\alpha^+)\ ]$.
x. Maximum: $[A] \vee [B] = [\ \max(a_\alpha^-, b_\alpha^-),\ \max(a_\alpha^+, b_\alpha^+)\ ]$.

Order Relations: An interval $[A]$ is said to be less than or equal to $[B]$, denoted by $[A] \preceq [B]$, if $a_\alpha^- \le b_\alpha^-$ and $a_\alpha^+ \le b_\alpha^+$,
or equivalently, $a_\alpha^- + a_\alpha^+ \le b_\alpha^- + b_\alpha^+$. Furthermore, $[A]$ is preferred to $[B]$ if $a_\alpha^- \ge b_\alpha^-$ and $a_\alpha^+ \ge b_\alpha^+$.

Interval Representation: The closed interval approximation of a PQFN $\tilde{A}^{PQFN}$is defined as $[A] = [a_\alpha^L, a_\alpha^U] = \{\, a \in \mathbb{R} \mid a_\alpha^L \le a \le a_\alpha^U \,\}$.

Center–Width Form: An interval can alternatively be expressed in terms of its center and width as $\langle A \rangle = \langle a_\alpha^C, a_\alpha^W \rangle$,
Where $a_\alpha^C = \frac{a_\alpha^L + a_\alpha^U}{2}, a_\alpha^W = \frac{a_\alpha^U - a_\alpha^L}{2}$.

Thus, the interval can be written as $\{\, a \in \mathbb{R} \mid a_\alpha^C - a_\alpha^W \le a \le a_\alpha^C + a_\alpha^W \,\}$.

Remark on Fuzzy Number Space: Let $\mathcal{F}(\mathbb{R})$ denote the set of all bounded and closed fuzzy numbers defined on $\mathbb{R}$. A function $f \in \mathcal{F}(\mathbb{R})$ satisfies:

xi. There exists at least one $x \in \mathbb{R}$such that $f(x) = 1$ (normality),
xii. For every $\alpha \in (0,1]$, the α-cut $f_\alpha = [f_\alpha^-, f_\alpha^+]$ forms a closed interval. It is evident that the set of real numbers $\mathbb{R}$is a subset of $\mathcal{F}(\mathbb{R})$.

3. Problem formulation and solution concepts

A general LFP problem (LFPP) can be expressed as follows:

$$\max \quad F(x) = \frac{P(x)}{Q(x)} = \frac{c^T x + \eta}{d^T x + \zeta}$$

subject to

$$x \in \Delta = \{x \in \mathbb{R}^n \mid Ax - b \le 0,\ x \ge 0\} \tag{1}$$

Here, $c, d \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, and $\eta, \zeta \in \mathbb{R}$.

To solve the LFPP, Charnes and Cooper introduced a transformation technique that converts the fractional objective into a linear form. This is achieved through the substitution $y = \gamma x$, where $\gamma$is a scalar variable. Using this transformation, the original LFPP can be reformulated into equivalent linear programming problems.

It has been shown that when the LFPP admits a finite optimal solution, only one of the resulting linear programming models needs to be solved. The appropriate formulation depends on the sign of the denominator of the objective function, as discussed by Zinots [22].

Furthermore, if there exists a feasible point $x \in \Delta$ such that $P(x) \geq 0$ and $Q(x) > 0$, then problem (1) can be transformed into the following equivalent optimization problem [23]: $\max \gamma P\left(\frac{y}{\gamma}\right)$ subject to

$$\gamma\, h\left(\frac{y}{\gamma}\right) \leq 0, \gamma\, Q\left(\frac{y}{\gamma}\right) \leq 1, y \geq 0, \gamma \geq 0 \tag{2}$$

On the other hand, if $P(x) < 0$, then the equivalent formulation of problem (1) is given by: $\max \gamma Q\left(\frac{y}{\gamma}\right)$ subject to

$$\gamma\, h\left(\frac{y}{\gamma}\right) \leq 0, -\gamma\, P\left(\frac{y}{\gamma}\right) \leq 1, y \geq 0, \gamma \geq 0 \tag{3}$$

Theorem 1. [24] Consider the LFPP $\max F(x) = \frac{P(x)}{Q(x)}$ subject to

$$x \in \Omega = \{x \in \mathbb{R}^n \mid h(x) \leq 0,\ x \geq 0\} \tag{4}$$

Assume that there exists $\gamma > 0$such that $\gamma = \frac{1}{Q(x)} > 0$. Then, the above problem can be equivalently transformed into the following parametric optimization problem: $\max \quad \gamma P(x)$ subject to:

$$(x, \gamma) \in \Upsilon = \left\{(x, \gamma) \mid h(x) \leq 0,\ x \geq 0,\ 0 < \gamma \leq \frac{1}{Q(x)}\right\} \tag{5}$$

Suppose that there exists a point $\varsigma \in \Omega$ such that $P(\varsigma) \geq 0$. If the original problem (4) attains a global optimal solution at $\bar{x}$, then the transformed problem (5) also achieves its global optimum at $(\bar{x}, \bar{\gamma})$, where $\bar{\gamma} = \frac{1}{Q(\bar{x})}$. Moreover, the optimal values of both formulations are identical.

Theorem 2. [24] Consider the LFPP defined in (4) $\max F(x) = \frac{P(x)}{Q(x)}, x \in \Omega$. Suppose that for every $\varsigma \in \Omega$, it holds that $P(\varsigma) < 0$. Then, the above problem can be equivalently reformulated as the following optimization problem:

$\max \gamma P(x)$
subject to

$$(x, \gamma) \in \Upsilon = \left\{(x, \gamma) \mid h(x) \leq 0,\ x \geq 0,\ \gamma \geq \frac{1}{Q(x)},\ \gamma > 0\right\} \tag{6}$$

If the original problem (4) attains its global maximum at $\bar{x}$, then the transformed problem (6) also achieves its global optimal solution at $(\bar{x}, \bar{\gamma})$, where $\bar{\gamma} = \frac{1}{Q(\bar{x})}$. Furthermore, the optimal objective values of both formulations are equal.

Definition 6. [25] The LFPP given in (1) is referred to as a standard concave–convex FPP (SCCFPP) if the following conditions are satisfied:

i. The function $P(\cdot)$is concave over the feasible region Δ, and there exists at least one point $\varsigma \in \Delta$such that $P(\varsigma) \geq 0$,
ii. The function $Q(\cdot)$is convex and strictly positive throughout the domain Δ.

Theorem 3. [24] Assume that problem (1) satisfies the conditions of a standard concave–convex FPP (SCCFPP). If the original problem achieves its global maximum at $\bar{x}$, then the transformed problem given in (3) also attains the same optimal value at the point $(\bar{x}, \bar{\gamma})$, where $\bar{\gamma} = \frac{1}{Q(\bar{x})}$.

Furthermore, the reformulated problem (3) possesses a concave objective function defined over a convex feasible region, ensuring favorable properties for optimization.

Proposition 1. [24] Suppose that $d_j > 0$for all $j = 1,2,\dots,n$ and $\zeta > 0$. Consider the fractional function

$$F(x) = \frac{c^T x + \eta}{d^T x + \zeta}, x \geq 0.$$

Then, the maximum and minimum values of $F(x)$over the nonnegative domain are given by

$$\bar{F} = \max\left\{\frac{c_j}{d_j}, \frac{\eta}{\zeta} \mid j = 1,2,\dots,n\right\},$$

and

$$\underset{\bar{}}{F} = \min\left\{\frac{c_j}{d_j}, \frac{\eta}{\zeta} \mid j = 1,2,\dots,n\right\}.$$

Theorem 4. [26] Let $\bar{F} = \frac{P(\bar{x})}{Q(\bar{x})} = \max\left\{\frac{P(x)}{Q(x)} \mid x \in \Delta\right\}$. Then, the value $\bar{F}$is optimal if and only if it satisfies the following equivalent condition: $\max_{x\in\Delta}\{P(x) - \bar{F}\, Q(x)\} = 0$, with the maximum attained at $x = \bar{x}$.

A transportation model with a linear fractional objective under PQF information can be expressed as follows:

$$\max \quad \tilde{Z}^{PQFN}(x) = \frac{\tilde{f}^{PQFN}(x)}{\tilde{g}^{PQFN}(x)} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{p}_{ij}^{PQFN}\ x_{ij} \oplus \tilde{\zeta}_0^{PQFN}}{\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{q}_{ij}^{PQFN}\ x_{ij} \oplus \tilde{\eta}_0^{PQFN}}$$

subject to the classical transportation constraints:

$$\begin{cases} \sum_{i=1}^{m} x_{ij} = a_j, \\ \sum_{j=1}^{n} x_{ij} = b_i, \\ x_{ij} \geq 0, \forall\ i,j \end{cases} \quad (7)$$

Here, the parameters $\tilde{p}_{ij}^{PQFN}$, $\tilde{q}_{ij}^{PQFN}$, $\tilde{\zeta}_0^{PQFN}$, and $\tilde{\eta}_0^{PQFN}$ are represented by PQFNs, capturing uncertainty in costs, profits, and other system parameters. By employing the closed interval approximation of PQFNs, the above fuzzy model can be transformed into an interval-valued formulation:

$$\max \quad \tilde{Z}^{CIA}(x) = \frac{\tilde{f}^{CIA}(x)}{\tilde{g}^{CIA}(x)} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n}[p_{ij}]\, x_{ij} \oplus [\zeta_0]}{\sum_{i=1}^{m} \sum_{j=1}^{n}[q_{ij}]\, x_{ij} \oplus [\eta_0]}$$

Subject to the same set of constraints given in (7).

In this formulation, the interval-valued parameters are defined as:

$$\begin{cases} [p_{ij}] = [p_{ij}^{\alpha^-}, p_{ij}^{\alpha^+}], [q_{ij}] = [q_{ij}^{\alpha^-}, q_{ij}^{\alpha^+}], \\ [a_j] = [a_j^{\alpha^-}, a_j^{\alpha^+}], [b_i] = [b_i^{\alpha^-}, b_i^{\alpha^+}] \\ [\zeta_0], [\eta_0] \in \mathcal{P}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R}) \end{cases} \quad (8)$$

Where each interval represents the closed approximation corresponding to the associated PQFN.

Definition 7. A vector $x = \{x_{ij}\}$, is said to be a feasible solution of problem (8) if it satisfies all the associated constraints. Let $\mathcal{G}$ denote the set of all such feasible solutions. A feasible point $x^* = \{x_{ij}^*\} \in \mathcal{G}$ is called an optimal solution of problem (8) if it satisfies

$$\frac{\sum_{i=1}^{m} \sum_{j=1}^{n} [p_{ij}]\, x_{ij}^*}{\sum_{i=1}^{m} \sum_{j=1}^{n} [q_{ij}]\, x_{ij}^*} \succeq \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} [p_{ij}]\, \bar{x}_{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n} [q_{ij}]\, \bar{x}_{ij}}, \forall\ \bar{x} = \{\bar{x}_{ij}\} \in \mathcal{G},$$

where the symbol $\succeq$represents the order relation defined on interval-valued quantities.

To solve problem (8), a variable transformation inspired by the Charnes–Cooper technique is employed. Let

$$\varsigma = \frac{1}{\tilde{g}^{CIA}(x)} = \frac{1}{\sum_{i=1}^{m} \sum_{j=1}^{n} [q_{ij}]\, x_{ij} \oplus [\eta_0]}.$$

Then, the normalization condition can be written as

$$\sum_{i=1}^{m} \sum_{j=1}^{n} [q_{ij}]\, x_{ij}\, \varsigma \oplus [\eta_0]\, \varsigma = 1.$$

Using this transformation, problem (8) is converted into the following equivalent form:

$$\max \quad \tilde{Z}^{CIA}(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} [p_{ij}]\, x_{ij}\, \varsigma \oplus [\zeta_0]\, \varsigma \text{ subject to}$$

$$\begin{aligned} &\sum_{i=1}^{m} x_{ij}\ \varsigma - a_j\, \varsigma = 0, \\ &\sum_{j=1}^{n} x_{ij}\ \varsigma - b_i\, \varsigma = 0 \\ &\sum_{i=1}^{m} \sum_{j=1}^{n} [q_{ij}]\, x_{ij}\, \varsigma \oplus [\eta_0]\, \varsigma = 1, x_{ij} \geq 0, \varsigma \geq 0. \end{aligned} \quad (9)$$

By Introduce a new variable $y_{ij} = x_{ij}\, \varsigma, \forall\ i, j$. Then, the model can be rewritten as:

$$\max \quad \tilde{Z}^{CIA}(y) = \sum_{i=1}^{m} \sum_{j=1}^{n} [p_{ij}]\, y_{ij} \oplus [\zeta_0]\, \varsigma$$

subject to

$$\begin{aligned} &\sum_{i=1}^{m} y_{ij} - a_j\, \varsigma \approx 0, j = 1,2, \dots, n, \\ &\sum_{j=1}^{n} y_{ij} - b_i\, \varsigma \approx 0, i = 1,2, \dots, m \\ &\sum_{i=1}^{m} \sum_{j=1}^{n} [q_{ij}]\, y_{ij} \oplus [\eta_0]\, \varsigma \approx 1, y_{ij} \geq 0, \varsigma \geq 0. \end{aligned} \quad (10)$$

Definition 8. [27] Let $\bar{x} = \{\bar{x}_{ij}\} \in S$, where Sdenotes the feasible region of problem (10). The point $\bar{x}$is called a solution to the problem (10) if and only if there does not exist another feasible point $x = \{x_{ij}\} \in S$such that $Z^{(-,+)}(\bar{x}) \prec^{(-,+)} Z(x)$ or $Z^{(C,W)}(\bar{x}) \prec^{(C,W)} Z(x)$, where $\prec^{(-,+)}$and $\prec^{(C,W)}$ denote dominance relations defined on interval-valued objective functions based on lower–upper bounds and center–width representations, respectively. In other words, a solution is one that is non-dominated with respect to the adopted interval comparison criteria.

Equivalent Multi-Objective Formulation

The set of solutions for problems (10) can be characterized as the set of Pareto-optimal solutions of the following multi-objective optimization model:

$$\begin{cases} \max\ Z^{-}(y) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}^{\alpha^-}\ y_{ij} + \zeta_0^-\ \varsigma \\ \max\ Z^{C}(y) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{p_{ij}^{\alpha^-}+p_{ij}^{\alpha^+}}{2}\ y_{ij} + \frac{\zeta_0^-+\zeta_0^+}{2}\ \varsigma \end{cases}$$

subject to the constraints:

$$\begin{cases} \sum_{i=1}^{m} y_{ij} - a_j\,\varsigma = 0, j = 1,2,\dots,n \\ \sum_{j=1}^{n} y_{ij} - b_i\,\varsigma = 0, i = 1,2,\dots,m \end{cases} \tag{11}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} q_{ij}^{\alpha^-}\ y_{ij} + \eta_0^-\ \varsigma \le 1, \text{ and}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} q_{ij}^{\alpha^+}\ y_{ij} + \eta_0^+\ \varsigma \ge 1, y_{ij} \ge 0, \varsigma \ge 0, \forall\ i,j.$$

4. Solution Procedure

The following steps outline the methodology for solving the multi-objective formulation given in problem (11):

Step 1: Begin with the original linear fractional transportation model defined in problem (7), where the parameters are expressed as PQFNs.

Step 2: Apply the closed interval approximation technique to convert the PQFN-based model into its corresponding interval-valued representation, resulting in problem (8).

Step 3: Determine the extreme values of the interval-based objective functions. Specifically, compute the lower-bound and center-based extrema: $Z_{\max}, Z_{\min}, Z_{\max}, Z_{\min}$.

Step 4: Evaluate the relative importance (weights) associated with the lower-bound and center-based objectives using the normalization scheme:

$$\begin{cases} w^{-} = \frac{Z_{\max}-Z_{\min}}{(Z_{\max}-Z_{\min})+(Z_{\max}-Z_{\min})}, w^{C} = \frac{Z_{\max}-Z_{\min}}{(Z_{\max}-Z_{\min})+(Z_{\max}-Z_{\min})} \end{cases} \tag{12}$$

Step 5: Construct a single-objective optimization model by aggregating multiple objectives through a weighted sum approach: $\max\ (w^{-}Z^{-}+w^{C}Z^{C})$ subject to the constraints defined in problem (11), along with:

$w^{-} \geq 0, w^{C} \geq 0, w^{-} + w^{C} = 1$ (13)

Step 6: Solve the resulting optimization problem using appropriate computational software such as LINGO 20.0 to obtain the optimal solution.

5.Numerical Example [28]

Step 1: Problem Description. Consider the following piecewise quadratic fuzzy LFTP (PQFLFTP):

$$\max \quad \tilde{Z}^{PQFN}(x) = \frac{\tilde{f}^{PQFN}(x)}{\tilde{g}^{PQFN}(x)} = \frac{\sum_{i=1}^{3}\sum_{j=1}^{4} \tilde{p}_{ij}^{PQFN} x_{ij} \oplus \tilde{\zeta}_0^{PQFN}}{\sum_{i=1}^{3}\sum_{j=1}^{4} \tilde{q}_{ij}^{PQFN} x_{ij} \oplus \tilde{\eta}_0^{PQFN}}$$

subject to:

$$\begin{cases} \sum_{i=1}^{3} x_{ij} = a_j, j = 1,2,3,4 \\ \sum_{j=1}^{4} x_{ij} = b_i, i = 1,2,3 \\ x_{ij} \geq 0 \end{cases} \quad (14)$$

The supply and demand vectors are: $(a_1, a_2, a_3) = (9,20,17), (b_1, b_2, b_3, b_4) = (7,9,14,16)$. All coefficients $\tilde{p}_{ij}^{PQFN}$ and $\tilde{q}_{ij}^{PQFN}$ are given as PQFNs.

Step 2: Interval Approximation. Using the closed interval approximation, the fuzzy model is transformed into the interval-valued FTP $\max \ Z^{CIA}(x) = \frac{f^{CIA}(x)}{g^{CIA}(x)}$, where the numerator and denominator are expressed using interval coefficients.

The resulting deterministic constraints remain:

$$\begin{cases} x_{11} + x_{12} + x_{13} + x_{14} = 9 \\ x_{21} + x_{22} + x_{23} + x_{24} = 20 \\ x_{31} + x_{32} + x_{33} + x_{34} = 17 \\ x_{11} + x_{21} + x_{31} = 7 \\ x_{12} + x_{22} + x_{32} = 9 \\ x_{13} + x_{23} + x_{33} = 14 \\ x_{14} + x_{24} + x_{34} = 16 \\ x_{ij} \geq 0 \end{cases} \quad (15)$$

Applying the Charnes–Cooper transformation with $y_{ij} = x_{ij}\varsigma$, the problem is converted into a linear form.

Step 3: Multi-Objective Reformulation. The transformed model is expressed as a bi-objective optimization problem:

$$\begin{aligned} \max \quad & Z^{-}(y) = y_{11} + 4y_{12} + 5y_{13} + 4y_{14} + 8y_{22} + y_{23} + 3y_{24} + 6y_{31} + 7y_{32} + 2y_{33} + 3y_{34}, \\ \max \quad & Z^{C}(y) = 3y_{11} + 5y_{12} + 6.5y_{13} + 5.5y_{14} + 1.5y_{21} + 10y_{22} + 3y_{23} + 4.5y_{24} + 7.5y_{31} \\ & \quad + 8.5y_{32} + 3.5y_{33} + 5.5y_{34}, \end{aligned}$$

subject to the transformed constraints:

Constraints (17) (balance and normalization constraints), $y_{ij} \geq 0, \varsigma \geq 0$.

Step 4: Determination of Extreme Values. The individual objective extrema are computed as:

$$Z_{\max} = 1.1901, Z_{\min} = 0.3218,$$
$$Z_{\max} = 1.8345, Z_{\min} = 0.5158.$$

Step 5: Weight Calculation. Using the normalization formula:

$$w^{-} = 0.60297, w^{C} = 0.39703.$$

Step 6: Aggregated Optimization Model.

The multi-objective model is reduced to a single-objective problem max $(w^{-}Z^{-}+w^{C}Z^{C})$, subject to the constraints in (17).

Step 7: Optimal Solution

Solving the above model (e.g., using LINGO), the optimal solution is obtained as:

$$y_{13} = \frac{9}{142}, y_{21} = \frac{7}{124}, y_{22} = \frac{4}{71}, y_{23} = \frac{5}{142},$$
$$y_{32} = \frac{1}{142}, y_{34} = \frac{8}{71}, \varsigma = \frac{1}{142}.$$

The corresponding optimal value is $Z^{*} = 1.44592$, which demonstrates an improvement over previously reported result.

Step 8: Back Transformation. Using $y_{ij} = x_{ij}\varsigma$, the optimal transportation plan is:

$x_{13} = 9, x_{21} = 7, x_{22} = 4, x_{23} = 5, x_{32} = 1, x_{34} = 8$, Which is better than the obtained in [28].

Step 9: Final Fuzzy Optimal Value. The optimal value in terms of PQFNs is $\tilde{Z}^{PQFN} = (50,\ 113,\ 175,\ 240,\ 274)$.

6. The Advantages and limitations of the study

Advantages

Enhanced modeling of uncertainty: The use of fully PQFNs allows for a more flexible and accurate representation of uncertainty compared to traditional triangular or trapezoidal fuzzy numbers.

Systematic transformation to solvable models: The closeness-based interval-valued approximation effectively converts complex fuzzy data into tractable interval forms, making the problem easier to handle computationally.

Utilization of classical optimization techniques: By applying the Charnes–Cooper transformation, the proposed approach converts the linear fractional model into an equivalent linear transportation problem (LTP), enabling the use of well-established solution methods.

Incorporation of decision-maker preferences: The use of order relations based on interval characteristics (lower bound, upper bound, midpoint, and width) provides a realistic and flexible framework for reflecting different risk attitudes and preferences.

Multi-objective optimization capability: The methodology accommodates multiple criteria simultaneously, improving the quality and applicability of solutions in real-world decision-making scenarios.

Efficient solution generation: The weighting method facilitates the identification of efficient (Pareto-optimal) solutions in a structured and computationally manageable way.

Practical applicability: The inclusion of a numerical example demonstrates the feasibility, effectiveness, and potential real-world applicability of the proposed approach.

Limitations

Increased computational complexity: The use of PQFNs and multi-stage transformations (fuzzy → interval → linear) may increase computational effort, especially for large-scale transportation problems.

Dependence on approximation accuracy: The closeness-based interval approximation may lead to some loss of information from the original fuzzy model, potentially affecting solution precision.

Subjectivity in preference modeling: The selection of order relations and weights depends on the decision maker’s judgment, which may introduce subjectivity and variability in the results.

Limited generalization to other fuzzy forms: The methodology is specifically designed for PQFNs and may not be directly applicable to other types of fuzzy numbers without modification.

Assumption of linear structure after transformation: The approach relies on transforming the problem into a linear model, which may oversimplify certain nonlinear characteristics inherent in real-world systems.

Single illustrative example: The validation is based on a limited numerical example, which may not fully capture the performance of the method across diverse or large-scale problem settings.

Weighting method limitations: The weighting approach may fail to capture all Pareto-optimal solutions, particularly in highly non-convex or complex multi-objective spaces.

7. Conclusions and Future Research Directions

This paper introduced a new model to solve the LFTP in a fuzzy setting with the objective function being a fully characterized PQFNs. In order to deal with the level of uncertainty, an approximation involving the use of the closeness was used to transform the PQFN-based model into an interval-valued LFTP. This was followed by the classical transformation suggested by Charnes and Cooper which was used to redefine the problem as an equivalent interval-valued linear transportation model. In order to deal with the interval character of the objective function, a multi-objective optimization methodology was created. This method uses order relations based on the preference of the decision maker taking into account critical interval properties such as lower bound, upper bound, midpoint and width. An efficient (Pareto-optimal) solution was then obtained by a weighted aggregation strategy. A numerical example was used to prove the effectiveness of the proposed methodology which performed better than the existing approaches.

Future Research Directions

Despite the fact that the proposed framework offers a systematic and efficient solution approach, there are a number of promising directions that can be pursued to make the methodology more applicable and robust:

1. The existing model can be extended to other uncertainty representations including intuitionistic FS, type-2 FS or neutrosophic sets, in order to represent more complex and higher-order uncertainty structures.
2. Future research can be done to establish better or hybrid interval approximation methods to reduce the loss of information related to the proximity-based interval representations of PQFNs.
3. The suggested methodology can be implemented on the actual, large-scale transportation and logistics systems, such as the supply chain systems, to test its computational effectiveness and applicability.
4. Integration of the proposed model with metaheuristic algorithms like genetic algorithms, particle swarm optimization or ant colony optimization can offer an effective solution to high-dimensional and complex problems.
5. It is possible to explore and compare other solution methods such as goal programming, ε-constraint method, and fuzzy goal programming with the weighted sum method to determine their performance.
6. The model can be generalized to dynamic or time-dependent transportation, and stochastic environments with uncertainty changing with time.
7. The sensitivity of the model can be investigated further with regard to change in weights, interval limits and preferences of the decision-maker to guarantee soundness and consistency of the solutions derived.

8. Other practical factors, including capacity constraints, environmental effects, and multi-modal transportation, can be incorporated in the model to make it more practical.

9. Developing an easy-to-use computation platform or decision support system would go a long way in enhancing the accessibility and feasibility of the proposed approach to practitioners and the industrial users.

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Conflicts of Interest

The authors declare no conflicts of interest.

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